Higher Category Theory – Lecture 1

Thesis: Quasicategories are a good model of $(\infty,1)$-categories.

The idea of $(\infty,1)$-categories

Definition: An n-category $\mathcal{C}$ is a collection of objects and

1-morphisms: $X \xrightarrow{f} Y$

2-morphisms:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

up to n-morphisms, between $(n-1)$-morphisms. K morphisms should be composable. Instead of associativity, we have a 2-morphism relating

$$(fg)h \underset{\text{2-morphism}}{\cong} f(gh)$$

where we write

$$\begin{tikzcd} & Z \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

and say h is a composite of f and g and α is a witness of this composition.

Definition: An (m,n)-category is a m-category where k-morphisms are invertible for $k > n$. Invertible in some weak sense: given a k-morphism $f: X \to Y$ $\exists$ $g: Y \to X$ and $fg \simeq 1_X$ and $gf \simeq 1_Y$ by some $(k+1)$-morphism, and so on.

Definition: An $(\infty,1)$-category C is an ∞-category such that 2-morphisms and higher are invertible.

Confusingly, in literature on $(\infty,1)$-categories, they are called "∞-categories".

Models

Let C be an $(\infty,1)$-category according to some definition. Let $C^{iso} \subseteq C$ be the subcat with only invertible 1-morphisms. Then C^{iso} is an $(\infty,0)$-category, so an ∞-grupoid.

Homotopy hypothesis: In any good definition of $(\infty,1)$-cat, the ∞-grupoids should model the homotopy types of spaces.

Definition: Two spaces $X, Y \in Top$ are weakly equivalent if there is a map $f: X \to Y$ such that

$$\pi_0(f): \pi_0(X) \to \pi_0(Y)$$

is a bijection and $\forall x \in X$

$$\pi_n(f): \pi_n(X,x) \to \pi_n(Y, f(x))$$

is a group isomorphism.

The goal of homotopy thy is to understand spaces $X \in Top$ up to weak equivalence. The weak equivalence class of X is called the homotopy type of X.

There are different models for homotopy types of spaces beyond $X \in Top$. The most famous alternative are simplicial sets. ∞-grupoids should also be models for homotopy types.

Given a definition of ∞-grupoids and a space X, there is an ∞-grupoid $\pi_{\leq\infty}(X)$ called the fundamental ∞-grupoid of X, with objects points of X, 1-morphisms paths in X, 2-morphisms homotopies between paths, and so on. There should be a map $\pi_{\leq\infty} : Top \to \infty\, Grpd$. The homotopy hypothesis says this is an equivalence, for a suitable notion of equivalence. We choose sSet as our model for ∞-grupoids, and define

$$Top \xrightarrow{\pi_{\leq\infty} = Sing.} sSet.$$

We can view ∞-grupoids $\subseteq$ sSets and Cat $\subseteq$ sSet. By comparing the two, we will find the definition of quasi-categories.

Definition: Let Δ be the cat with

- Objects: the finite totally ordered sets

$$[n] = \{0 < 1 < \ldots < n\}$$

for $n \in \mathbb{N}_0$;

- Morphisms: order preserving maps $f : [m] \to [n]$ s.t if $i \leq j$, then $f(i) \leq f(j)$.

Definition: A simplicial set is a contravariant functor

$$X : \Delta \to \mathrm{Set}.$$

We turn spaces $X \in \mathrm{Top}$ by the singular functor

$$\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{sSet}$$

$$X \mapsto \mathrm{Sing}(X)$$

where $\mathrm{Sing}(X)[n] = \mathrm{Sing}_n(X) = \mathrm{Map}(\Delta^n, X)$ for

$$\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1 \}.$$

There is a functor $N_\bullet : \mathrm{Cat} \to \mathrm{sSet}$ where

$$N_n C = \mathrm{Fun}([n], C).$$

This functor is called the nerve. Graphically

$$[n] = \{0 < 1 < \dots < n\}$$

$$= \{0 \to 1 \to \dots \to n\}.$$

so

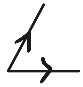
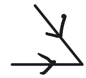
$$\mathrm{Fun}([n], C) = \{ X_0 \xrightarrow{f_0} \dots \xrightarrow{f_{n-1}} X_n : X_i \in C, f_i : X_i \to X_{i+1} \}$$

$$= n\text{-tuples of composable arrows.}$$

We want to characterize $\mathrm{Sing}_\bullet(X)$ and $N_\bullet C$, but first $N_\bullet G$, the nerve of a grupoid.

Definition: The k-th horn of the n-simplex Δ^n

$$\Lambda^n_k = \partial\Delta^n \setminus \text{k-th face}.$$

Example: $\Lambda^2_0 =$  , $\Lambda^2_1 =$  , $\Lambda^2_2 =$  .

Definition: A k-horn of dimension n in X is a map

$$\Lambda^n_k \longrightarrow X$$

and this horn has filler if there is a lift

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur] & \end{tikzcd}$$

Theorem: For $X = N_\bullet G$ the nerve of a grupoid, all horns have unique fillers.

Theorem: For $X = \mathrm{Sing}_\bullet(S)$, $S \in \mathrm{Top}$, all horns have fillers.

Theorem: For $X = N_\bullet C$, all inner horns have unique fillers. A horn is inner if $0 < k < n$.

Definition: A quasicategory C is a simplicial set where all inner horns have fillers.