

Higher Category Theory - Lecture 2

Notation:

- Action of simplicial operators

$$X_n \xrightarrow{X(f)} X_m$$

$$a \mapsto a.f$$

- Description of simplicial operators

$$\delta: [m] \rightarrow [n]$$

$$0 \mapsto \delta(0)$$

\vdots

$$m \mapsto \delta(m)$$

we write as $\delta = \langle \delta_0 \dots \delta_n \rangle$.

Example: Face operators: $d^i: [n] \rightarrow [m]$ "omits i ", so for

$$[0] \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} [1]$$

can be written as $d^0 = \langle 1 \rangle$ and $d^1 = \langle 0 \rangle$.

Example: Codegeneracy operators: $[0] \xleftarrow{s^0} [1] \xleftarrow{s^1} [2]$ "repeat i ", so they can be written as $s^i: [2] \rightrightarrows [1]$

$$s^0 = \langle 001 \rangle \text{ and } s^1 = \langle 011 \rangle.$$

We can shorten notation by writing $a.f = a_{i_1 \dots i_n}$.

The nerve of a category: Let \mathcal{C} be a category, we can define a sset

$$N_n \mathcal{C} = \text{Hom}_{\text{cat}}([n], \mathcal{C})$$

and given $f: [m] \rightarrow [n]$ we have

$$N_n(\mathcal{C}) \longrightarrow N_m(\mathcal{C})$$

$$a: [n] \rightarrow \mathcal{C} \mapsto a \circ f: [m] \rightarrow \mathcal{C}$$

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor, then

$$N_n(\mathcal{C}) \longrightarrow N_m(\mathcal{D})$$

$$a \mapsto F \circ a$$

is a simplicial map. In particular, we have

- $N_0 \mathcal{C} \simeq \text{Ob } \mathcal{C}$

- $N_1 \mathcal{C} \simeq \text{Mor } \mathcal{C}$ and the operators

$$\langle 0 \rangle: [0] \rightarrow [1] \quad \rightsquigarrow \quad f \langle 0 \rangle = \text{source } f$$

$$\langle 1 \rangle: [0] \rightarrow [1] \quad \rightsquigarrow \quad f \langle 1 \rangle = \text{target } f$$

$$\langle 00 \rangle: [1] \rightarrow [0] \quad \rightsquigarrow \quad \langle 00 \rangle = \text{id}_x$$

- $N_2 \mathcal{C} \simeq$ composable pairs (f, g) s.t. $f \langle 1 \rangle = g \langle 0 \rangle$.

Proposition: Let \mathcal{G} be a category with object set $\text{ob } \mathcal{G}$ and morphism set $\text{mor } \mathcal{G}$. We have

1) there is a bijective correspondence

$$N_n \mathcal{G} \xrightarrow{\sim} \{(g_1, \dots, g_n) \in (\text{mor } \mathcal{G})^n \mid \text{tar } g_{i-1} = \text{sor } g_i\}$$

$$a: [n] \rightarrow \mathcal{G} \mapsto (a_{0,1}, \dots, a_{n-1,n})$$

2) with respect to $N_n \mathcal{G} \xrightarrow{\sim} \text{mor } \mathcal{G} \times \dots \times \text{mor } \mathcal{G}$, the map

$$\delta^*: N_n \mathcal{G} \rightarrow N_m \mathcal{G}$$

induced by a simplicial operator $\delta: [m] \rightarrow [n]$ coincides with

$$(g_1, \dots, g_n) \rightarrow (h_1, \dots, h_m)$$

$$h_k = \begin{cases} \text{id} & \text{if } \delta(k-1) = \delta(k) \\ g_j g_{j-1} \dots g_{i+1} & \text{if } \delta(k-1) = i < j = \delta(k). \end{cases}$$

Proposition: A simplicial set X is isomorphic to the nerve of some category iff $\forall n \gg 2$

$$\phi_n: X_n \rightarrow \{(g_1, \dots, g_n) \in X_1^n \mid g_{i-1} \langle 1 \rangle = g_i \langle 0 \rangle\}$$

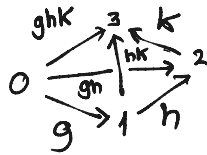
are bijections.

Proof: Let X satisfy the previous conditions. Then we define

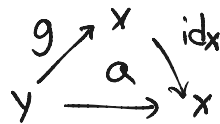
$$X_0 = \text{ob } \mathcal{G} \quad \text{and} \quad X_1 = \text{mor } \mathcal{G}$$

For identities we set $\langle 00 \rangle^*: X_0 \rightarrow X_1$. For $(g, h) \in X_1 \times_{X_0} X_1$, the composite is $a_{02} = \phi_2^{-1}(f, g) \langle 0, 2 \rangle$. Associativity: (g, h, k)

are associative, so we have $a = \phi_3^{-1}(g, h, k)$



Given $x \in X_0, g \in X_1$ s.t. $g \langle 1 \rangle = x$. (g, x_{00}) form a composable pair since $x_{00} = (x \langle 00 \rangle) \langle 0 \rangle = x \langle 00 \rangle \langle 0 \rangle = x$. We have a 2-cell



but $(g \langle 011 \rangle) \langle 01 \rangle = g \langle 011 \rangle \langle 01 \rangle = g$ and

$$\begin{aligned} (g \langle 011 \rangle) \langle 12 \rangle &= g \langle 011 \rangle \langle 12 \rangle = g \langle 11 \rangle \\ &= (g \langle 1 \rangle) \langle 00 \rangle = x_{00} = \text{id}_x \end{aligned}$$

and $(g \langle 011 \rangle) \langle 00 \rangle = g \langle 011 \rangle \langle 00 \rangle = g$, so by uniqueness of a , it follows that $a = \langle 011 \rangle$.

Proposition: $N_*: \text{Cat} \rightarrow \text{sSet}$

$$G \mapsto N_* G$$

gives $\text{Hom}_{\text{cat}}(G, D) \cong \text{Hom}_{\text{sset}}(N_* G, N_* D)$.

Definition: The horns $\Lambda_j^n \subset \Delta^n$ are defined by

$$(\Lambda_j^n)_k = \{f: [k] \rightarrow [n] \mid ([n] \setminus j) \not\subseteq f([k])\}.$$

A horn Λ_j^n is inner if $0 < j < n$.

Proposition: Let X be a simplicial set, X is the nerve of a category iff

$$\text{Hom}(\Delta^n, X) \xrightarrow{\sim} \text{Hom}(\Lambda_j^n, X)$$

for all $n \geq 2$, $0 < j < n$.