Higher Category Theory - Lecture 2

Notation:

- Action of simplicial operators

$$X_n \xrightarrow{X(f)} X_m$$

$$a \longmapsto a.f$$

- Description of simplicial operators

$$\begin{aligned} \delta : [m] &\longrightarrow [n] \\ 0 &\longmapsto \delta(0) \\ &\vdots \\ m &\longmapsto \delta(m) \end{aligned}$$

we write as $\delta = \langle \delta_0 \dots \delta_n \rangle$.

Example: Coface operators: $d^i : [n] \longrightarrow [m]$ "omits i", so for

$$[0] \overset{d^0}{\underset{d^1}{\rightrightarrows}} [1]$$

can be writen as $d^0 = \langle 1 \rangle$ and $d^1 = \langle 0 \rangle$.

Example: Codegeneracy operators: $[0] \xleftarrow{s^0} [1] \overset{s^0}{\underset{s^1}{\leftleftarrows}} [2]$ "repeat i", so they can be written as $s^i : [2] \rightrightarrows [1]$

$$s^0 = \langle 001 \rangle \quad \text{and} \quad s^1 = \langle 011 \rangle.$$

We can shorten notation by writting $a.f = a_{i_1 \dots i_n}$.

The nerve of a category: Let $\mathcal{C}$ be a category, we can define a sset

$$N_n\mathcal{C} = \mathrm{Hom}_{cat}([n],\mathcal{C})$$

and given $f:[m]\to[n]$ we have

$$N_n(\mathcal{C}) \longrightarrow N_m(\mathcal{C})$$

$$a:[n]\to\mathcal{C} \mapsto a.f:[m]\to\mathcal{C}$$

Let $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ be a functor, then

$$N_n(\mathcal{C}) \longrightarrow N_m(\mathcal{D})$$

$$a \mapsto \mathcal{F}\circ a$$

is a simplicial map. In particular, we have

- $N_0\mathcal{C} \approx \mathrm{Ob}\,\mathcal{C}$
- $N_1\mathcal{C} \approx \mathrm{Mor}\,\mathcal{C}$ and the operators

$$\begin{matrix} \langle 0\rangle : [0]\longrightarrow[1] & & f\langle 0\rangle = \text{source } f \\ & \rightsquigarrow & \\ \langle 1\rangle : [0]\longrightarrow[1] & & f\langle 1\rangle = \text{target } f \end{matrix}$$

$$\langle 00\rangle : [1]\to[0] \rightsquigarrow \langle 00\rangle = id_x$$

- $N_2\mathcal{C} \approx$ composable pairs (f,g) s.t $f\langle 1\rangle = g\langle 0\rangle$.

Proposition: Let $\mathcal{C}$ be a category with object set $\mathrm{ob}\,\mathcal{C}$ and morphism set $\mathrm{mor}\,\mathcal{C}$. We have

1) there is a bijective correspondence

$$N_n\mathcal{C} \xrightarrow{\sim} \{(g_1,\dots,g_n) \in (\mathrm{mor}\,\mathcal{C})^n \mid \mathrm{tar}\, g_{i-1} = \mathrm{sor}\, g_i\};$$

$$a:[n]\to\mathcal{C} \longmapsto (a_{01},\dots,a_{n-1,n})$$

2) with respect to $N_n\mathcal{C} \xrightarrow{\sim} \mathrm{mor}\,\mathcal{C}\times\dots\times\mathrm{mor}\,\mathcal{C}$, the map

$$\delta^*: N_n\mathcal{C} \to N_m\mathcal{C}$$

induced by a simplicial operator $\delta:[m]\to[n]$ coincides with

$$(g_1,\dots,g_n) \longrightarrow (h_1,\dots,h_m)$$

$$h_k = \begin{cases} \mathrm{id} & \text{if } \delta(k-1)=\delta(k) \\ g_j g_{j-1}\cdots g_{i+1} & \text{if } \delta(k-1)=i<j=\delta(k). \end{cases}$$

Proposition: A simplicial set X is isomorphic to the nerve of some category iff $\forall n \geqslant 2$

$$\phi_n: X_n \to \{(g_1,\dots,g_n)\in X_1^n \mid g_{i-1}\langle 1\rangle = g_i\langle 0\rangle\}$$

are bijections.

Proof: Let X satisfy the previous conditions. Then we define

$$X_0 = \mathrm{ob}\,\mathcal{C} \qquad \text{and} \qquad X_1 = \mathrm{mor}\,\mathcal{C}$$

For identities we set $\langle 00\rangle^*: X_0 \to X_1$. For $(g,h)\in X_1\times_{X_0} X_1$, the composite is $a_{02} = \phi_2^{-1}(f,g)\langle 0,2\rangle$. Associativity: (g,h,k)

are associative, so we have $a = \phi_3^{-1}(g, h, k)$

$$\begin{tikzcd}
 & & 3 & \\
0 \arrow[urr, "ghk"] \arrow[rrr, "gh"] \arrow[dr, "g"'] & & & 2 \arrow[ul, "k"'] \\
 & 1 \arrow[uur, "hk"] \arrow[urr, "h"'] & &
\end{tikzcd}$$

Given $x \in X_0$, $g \in X_1$ s.t $g\langle 1\rangle = x$. (g, x_{00}) form a composable pair since $x_{00} = (x\langle 00\rangle)\langle 0\rangle = x(\langle 00\rangle\langle 0\rangle) = x$. We have a 2-cell

$$\begin{tikzcd}
 & x \arrow[dr, "\mathrm{id}_x"] \arrow[d, phantom, "a"] & \\
y \arrow[ur, "g"] \arrow[rr] & {} & x
\end{tikzcd}$$

but $(g\langle 011\rangle)\langle 01\rangle = g(\langle 011\rangle\langle 01\rangle) = g$ and

$$(g\langle 011\rangle)\langle 12\rangle = g(\langle 011\rangle\langle 12\rangle) = g\langle 11\rangle$$
$$= (g\langle 1\rangle)\langle 00\rangle = x_{00} = \mathrm{id}_x$$

and $(g\langle 011\rangle)\langle 00\rangle = g(\langle 011\rangle\langle 00\rangle) = g$, so by uniqueness of a, it follows that $a = \langle 011\rangle$.

Proposition:

$$N_\bullet : \mathrm{Cat} \longrightarrow \mathrm{sSet}$$
$$\mathcal{C} \longmapsto N_\bullet \mathcal{C}$$

gives $\mathrm{Hom}_{\mathrm{cat}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathrm{sSet}}(N_\bullet \mathcal{C}, N_\bullet \mathcal{D})$.

Definition: The horn $\Lambda^n_j \subset \Delta^n$ are defined by

$$(\Lambda^n_j)_k = \{ f : [k] \to [n] \mid ([n] \setminus j) \not\subset f([k]) \}.$$

A horn Λ^n_j is inner if $0<j<n$.

Proposition: Let X be a simplicial set, X is the nerve of a category iff

$$\mathrm{Hom}(\Delta^n, X) \xrightarrow{\sim} \mathrm{Hom}(\Lambda^n_j, X)$$

for all $n \geq 2$, $0<j<n$.