Higher Category Theory - Lecture 3

Definition: An ∞-category is a simplicial set $\mathcal{C}$ such that every inner horn admits an (non-unique) extension.

Proposition: The product $X \times Y$ of ∞-cats is an ∞-cat.

Proposition: The coproduct $X \amalg Y$ of ∞-cats is an ∞-cat.

Definition: An ∞-subcategory is a sub-simplicial set $\mathcal{C}' \subseteq \mathcal{C}$ such that $\forall n \geqslant 2$ and $0 < k < n$, $f : \Delta^n \to \mathcal{C}$ such that $f(\Lambda^n_k) \subseteq \mathcal{C}'$, then $f(\Delta^n) \subseteq \mathcal{C}'$.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "f|_{\Lambda^n_k}"] \arrow[d] & \mathcal{C}' \arrow[d] \\
\Delta^n \arrow[r, "f"'] \arrow[ur, dashed] & \mathcal{C}
\end{tikzcd}$$

Example: Consider $\mathrm{Cat}_1 \subseteq \mathcal{C}at_1$. Notice that $N_\bullet(\mathrm{Cat}_1) \subseteq \mathcal{C}at_1$ as a sub-complex, but not as a ∞-subcategory.

Definition: A $\mathcal{C}' \subseteq \mathcal{C}$ is a full ∞-subcat if $\forall a \in \mathcal{C}_n$ we have that $a \in \mathcal{C}'_n$ if $a_i \in \mathcal{C}'_0$.

Definition: Consider the involution $op : \Delta \to \Delta$ given by

$$\begin{tikzcd}
{[n]} \arrow[r] \arrow[d, "f = \langle f_0, \dots, f_n \rangle"'] & {[n]} \arrow[d, "\langle m - f_1, \dots m - f_n \rangle"] \\
{[m]} \arrow[r] & {[m]}
\end{tikzcd}$$

Then $\mathcal{C} : \Delta^{op} \to \mathrm{Set}$ be a simplicial set. Then $\mathcal{C} \circ op$ defines the opposite category.

We have that $(\Delta^n)^{op} \simeq \Delta^n$, $(I^n)^{op} \simeq I^{op}$, and $(\Lambda^n_k)^{op} \simeq \Lambda^n_{n-k}$.

Definition: A functor between ∞-cats is a simplicial map $F: \mathcal{C} \to \mathcal{D}$. We define Cat_∞, whose objects are ∞-cats, and morphisms are functors of ∞-cats.

Proposition:

$$\left\{ \begin{array}{c} \text{Natural transformations} \\ Q: F \Rightarrow G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \quad\quad \text{Functors } \phi: \mathcal{C}[1] \to \mathcal{D} \\ \text{st } \phi|_{\mathcal{C}\times\{0\}}: \mathcal{C}\times\{0\} \to \mathcal{D} = F \\ \text{and } \phi|_{\mathcal{C}\times\{1\}}: \mathcal{C}\times\{1\} \to \mathcal{D} = G \end{array} \right\}$$

Definition: An ∞-natural transformation between functors F_0 and F_1 is a simplicial map $\phi: \mathcal{C}\times\Delta^1 \to \mathcal{D}$ such that $\phi|_{\mathcal{C}\times\{i\}} = F_i$.

Homotopy Category of ∞-categories

The fundamental category of a sset X is

i) A category hX ;

ii) $\alpha: X \to N.(hX)$ s.t for categories $\mathcal{C}$

$$\alpha^*: \mathrm{Hom}(N.(hX), N.\mathcal{C}) \longrightarrow \mathrm{Hom}(X, N.\mathcal{C})$$

is a bijection.

Proposition: We construct by generators and relations:

i) $\mathrm{ob}\, hX = X_0$

ii) $\mathrm{mor}\, hX = [a_n; \dots; a_1]$ where $[a] = [\,]_x$ and $[g,f] \sim [h]$ if $a_{0,1} = f$,

$a_{1,2}=g$ and $a_{0,2}=h$.

This construction gives a functor $h : sSet \to Cat$ s.t

$$\begin{tikzcd}
Hom_{cat}(hX, \mathcal{C}) \arrow[rr, "\sim"] \arrow[dr] & & Hom_{sSet}(X, N.\mathcal{C}) \\
& Hom_{sSet}(N.(hX), N.\mathcal{C}) \arrow[ur, "\alpha^*"'] &
\end{tikzcd}$$

Definition: Let $\mathcal{C}$ be an ∞-cat, and

$$hom_{\mathcal{C}}(x,y) = \{ f \in \mathcal{C}_1 \mid f_0 = x,\ f_1 = y \}.$$

We say $f \sim_\ell g$ iff $\exists\, a \in \mathcal{C}_2$ s.t

$$\begin{tikzcd}
x \arrow[dr, "f"] \arrow[dd, "1_x"'] & \\
 & y \\
x \arrow[ur, "g"'] \arrow[u, phantom, shift left=6, "a"] &
\end{tikzcd}$$

and $f \sim_r g$ iff $\exists\, b \in \mathcal{C}_2$ s.t

$$\begin{tikzcd}
 & y \arrow[dd, "1_y"] \\
x \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "b"] & {} \\
 & y
\end{tikzcd}.$$

Proposition: $f, g, h : x \to y$ in an ∞-cat, then

i) $f \sim_\ell f$;

ii) $f \sim_\ell g$ and $g \sim_\ell h \Rightarrow f \sim_\ell h$;

iii) $f \sim_\ell g \Rightarrow f \sim_r g$;

iv) $f \sim_r g \Rightarrow f \sim_\ell g$.

Therefore $\sim_{\ell/r}$ are equivalence relations.

Lemma: $f \approx f'$ and $g \approx g'$. Let h be a composite of (g, f) and h' a composite of (g', f'), then $h \approx h'$.

Lemma: $f: x \to y$ then $[f] \circ [id_x] = [f] = [id_y] \circ [f]$.

Lemma: If $[g] \circ [f] = [u]$ and $[h] \circ [g] = [v]$, then $[h] \circ [u] = [f] \circ [v]$.

Definition: Let $\mathcal{C}$ be an ∞-cat. We define $ob(h\mathcal{C}) = \mathcal{C}_0$ and $mor(h\mathcal{C}) := hom_{\mathcal{C}}(X, Y)/\approx$ where $\pi: \mathcal{C} \to N.(h\mathcal{C})$.

Proposition: Let $\mathcal{C}$ be an ∞-cat, $\mathcal{D}$ a cat, and let $\phi: \mathcal{C} \to N.\mathcal{D}$ be a simplicial map. Then there exists a unique map $\psi: N.(h\mathcal{C}) \to N.(\mathcal{D})$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] \arrow[dr, "\phi"'] & N.(h\mathcal{C}) \arrow[d] \\ & N.(\mathcal{D}) \end{tikzcd}.$$

Recall that for categories we have

$$Fun(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq Fun(\mathcal{C}, Fun(\mathcal{D}, \mathcal{E})).$$

We want a bijection

$$Hom_{sset}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq Hom_{sset}(\mathcal{C}, Hom_{sset}(\mathcal{D}, \mathcal{E}))$$

so we have to propose a sset structure on $Hom_{sset}(\mathcal{D}, \mathcal{E})$. We say

$$Fun(X, Y)_n := Hom_{sset}(\Delta^n \times X, Y)$$

$$\delta: [n] \to [m] \qquad Hom(\delta \times id, Y): Hom(\Delta^n \times X, Y) \to Hom(\Delta^m \times X, Y).$$

Proposition: $\mathrm{Fun}: sSet^{op} \times sSet \longrightarrow sSet$ and

$$\{\Delta^n \longrightarrow \mathrm{Fun}(X,Y)\} \longleftrightarrow \{\Delta^n \times X \longrightarrow Y\}.$$

Proposition $\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, \mathrm{Fun}(Y,Z))$ natural on all variables.

Proof: $\mathrm{Hom}(X \times Y, Z) \longrightarrow \mathrm{Hom}(X, \mathrm{Fun}(Y,Z))$

$$f: X \times Y \to Z \longmapsto \tilde{f}: X \to \mathrm{Fun}(Y,Z)$$

$$x \in X_n \longmapsto \tilde{f}(x)$$

$$\begin{tikzcd} \Delta^n \times Y \arrow[r, "x \times id"] \arrow[dr] & X \times Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

and we construct inverse $g: \mathrm{Hom}(X, \mathrm{Fun}(Y,Z)) \longrightarrow \mathrm{Hom}(X \times Y, Z)$

(incomplete!)