

MAIN GOALS

Things we will prove today:

- $\text{Fun}(X, \mathcal{C})$ is an ∞ -category if \mathcal{C} is an ∞ -category.
- $\text{Comp}_{\mathcal{C}}(f, g)$ is a contractible Kan complex iff \mathcal{C} is an ∞ -category.
- $\text{Map}_{\mathcal{C}}(x, y)$ is an ∞ -category (unsatisfactory, but we can't do more now...)

but we will prove many more things!

Outline

1. Introduce weakly saturated classes and inner-anodyne maps

2. Introduce:

2.1 Lifting Calculus

2.2 Inner Fibrations

3. Digression on Inner Fibrations

4. A tourist's view of:

4.1 Small Object Argument & Weak Factorization

4.2 Skeletal Filtrations

4.3 Trivial Fibrations

5. Pushout-Product & Pullback-Horn
(a.k.a enriched lifting)

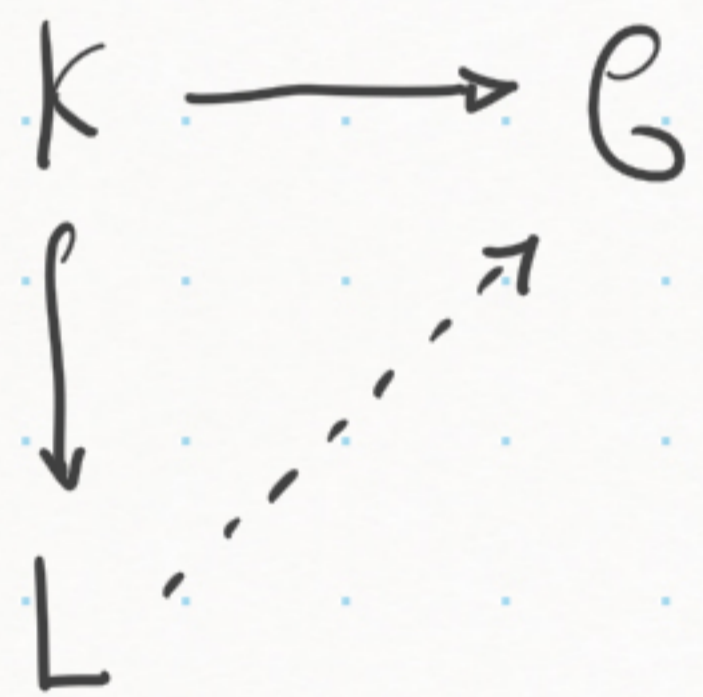
6. Function Complexes of ∞ -categories are ∞ -categories

7. Sanity check: $\mathcal{C}^{\Delta^2} \longrightarrow \mathcal{C}^{\Lambda^2}$
does what we want...

Weakly Saturated Classes and Inner-Anodyne Maps

• ∞ -categories are defined by an "extension property"

• \mathcal{C} is an ∞ -Category if



for every $K \in L \in \{ \Delta_j^n \subseteq \Delta^n : n \geq 2, 0 < j < n \}$

Q: Is $\{ \Delta_j^n \subseteq \Delta^n : n \geq 2, 0 < j < n \}$ precisely the class of morphisms with this extension property?
 not really

Weakly Saturated Classes: A class of morphisms

Such that:

1) Contains all **isomorphisms** $f: X \xrightarrow{\sim} Y$

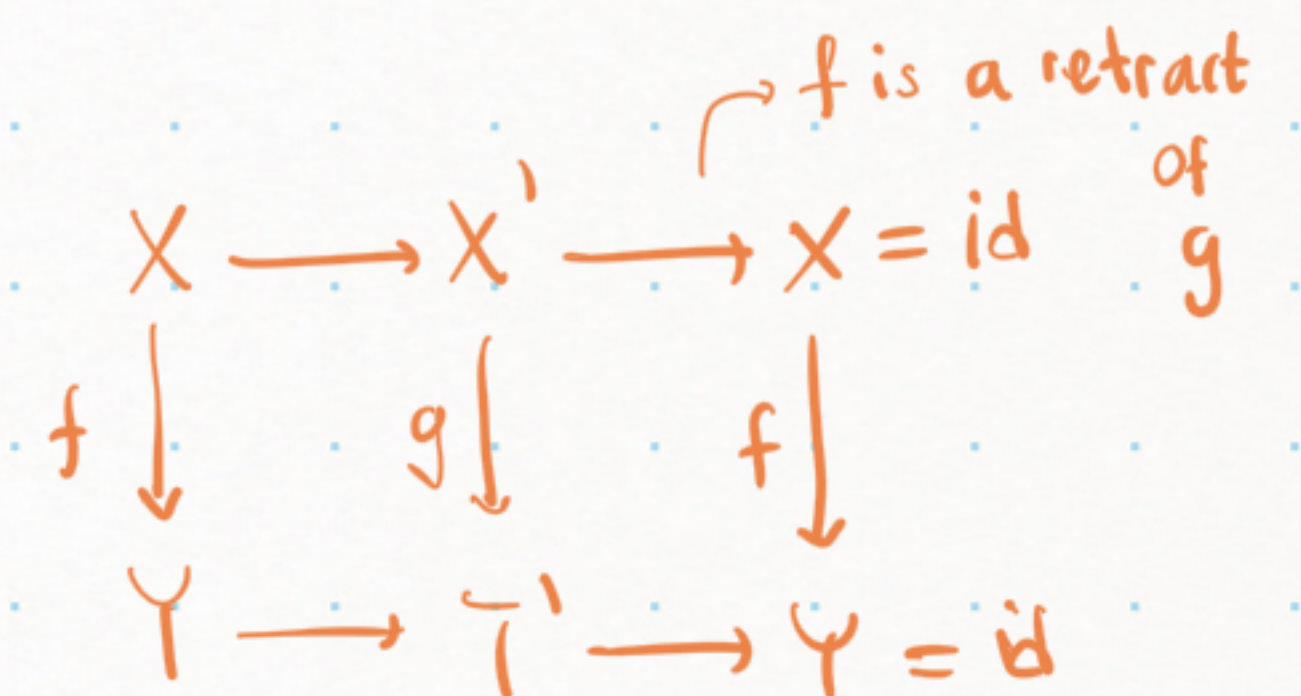
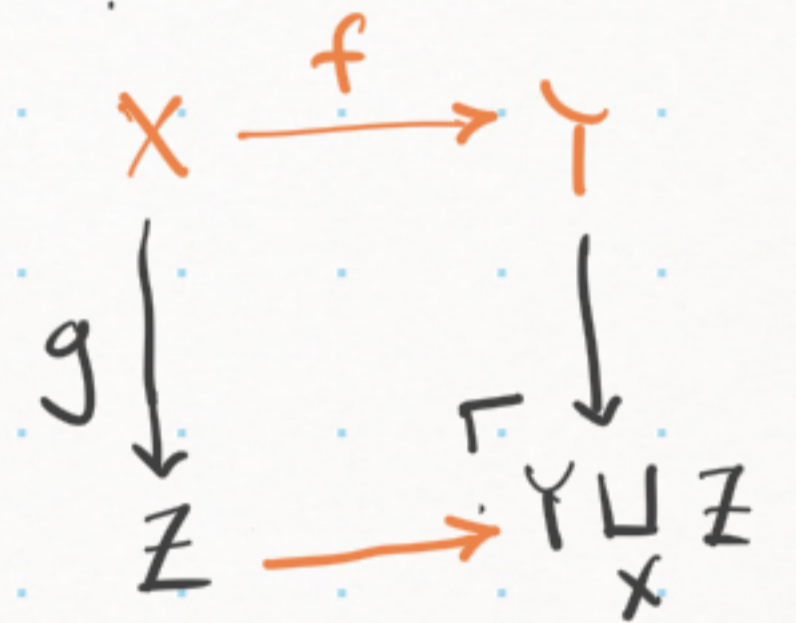
2) is closed under **base change**

3) is closed under **composition**

4) is closed under **transfinite composition**

5) is closed under **coproducts**

6) is closed under **retracts**



Def: Given a class of maps S , we define its weak saturation to be the smallest saturated class containing S .

$$S \rightsquigarrow \bar{S}$$

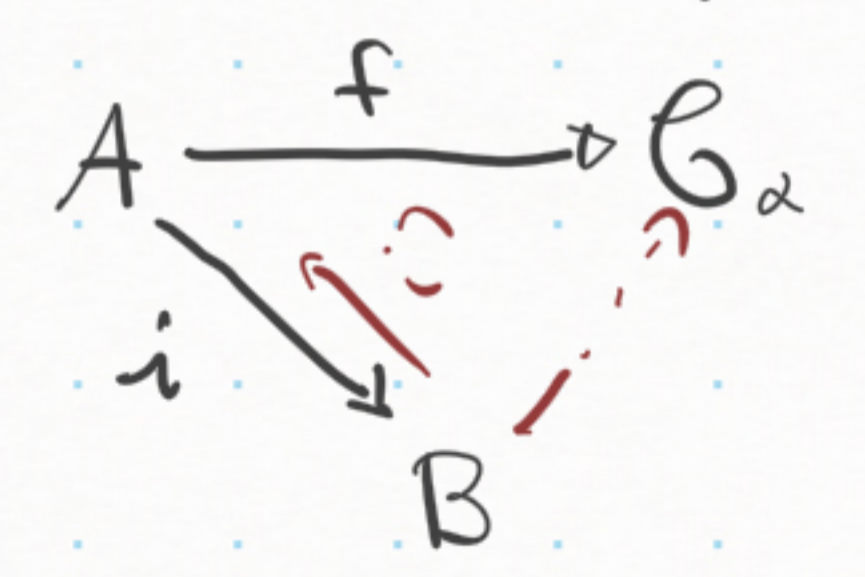
Examples: $\overline{\{ \{0,1\} \rightarrow \{1\} \}} = \text{Surj}$
 $\overline{\{ \phi: \{1\} \rightarrow \{1\} \}} = \text{Mono}$

Prop: Consider a collection $\{C_\alpha\}$ of simplicial sets. Really crucial result!

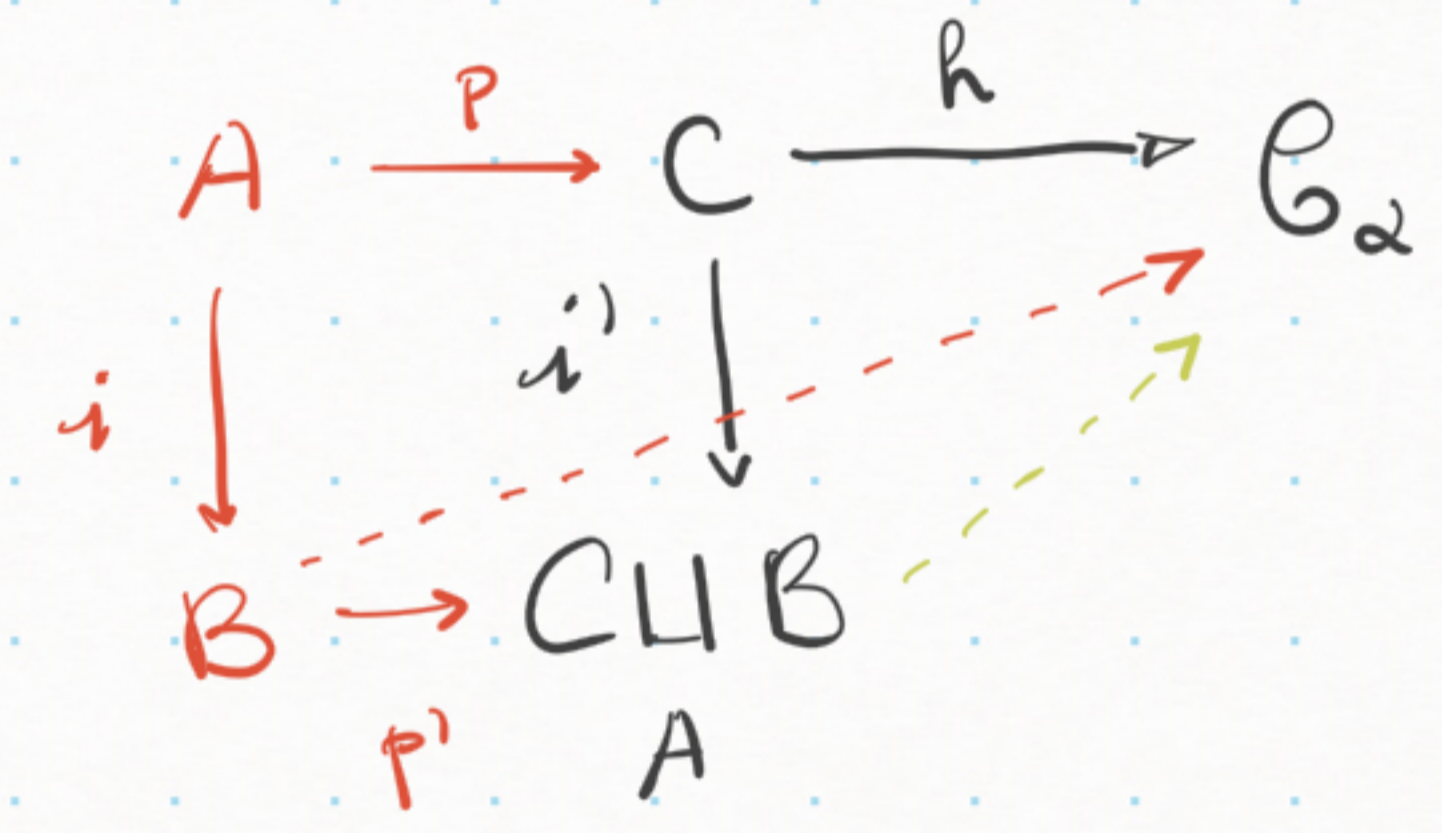
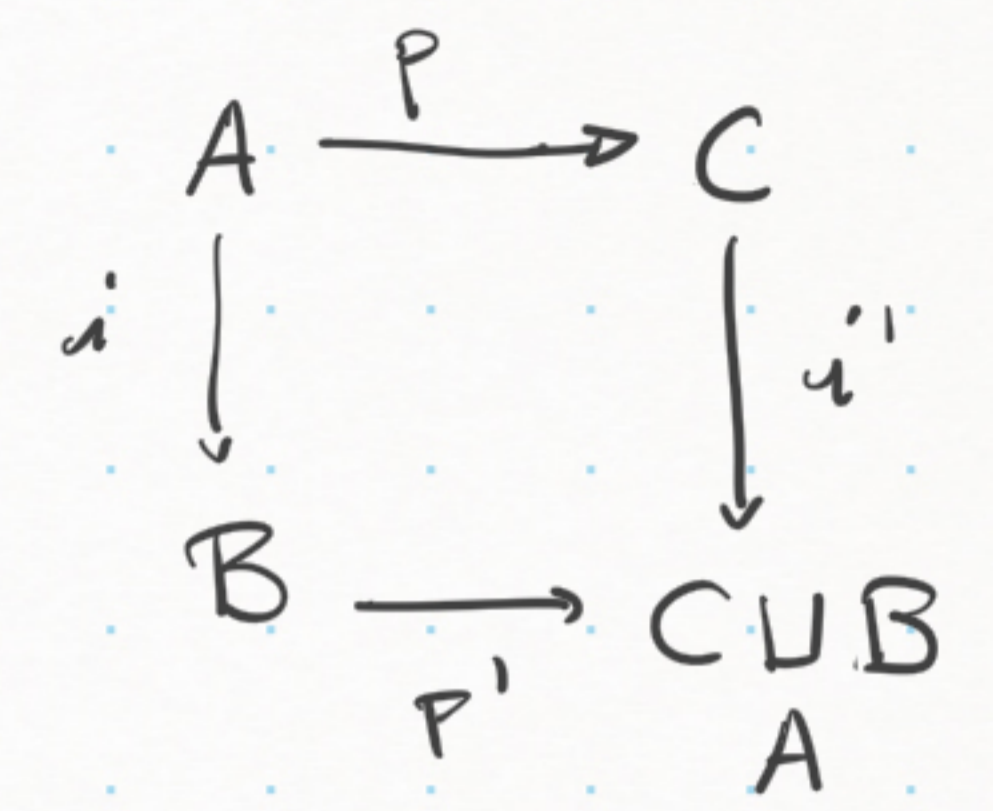
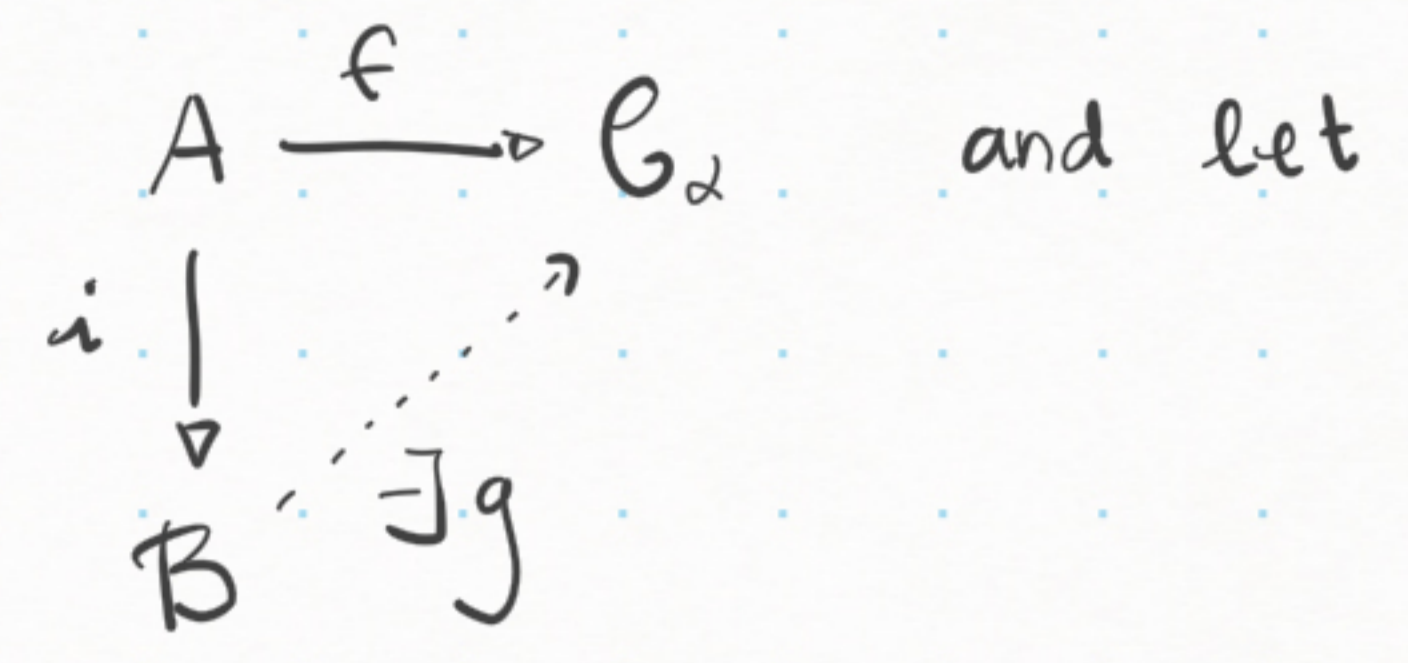
$$\text{Let } A := \left\{ i: A \rightarrow B \mid \begin{array}{c} A \xrightarrow{f} C_\alpha \\ \downarrow i \quad \nearrow \exists g \\ B \end{array} \forall f \in \text{Hom}(A, C_\alpha) \forall \alpha \right\}$$

Then A is a weakly saturated class

Proof: We must show that this class
 1) contains all isomorphisms: $\xrightarrow{\text{Assume that } A \xrightarrow{i} B \text{ is an isomorphism}}$



2) is closed under cobase change



We could have defined a class of weakly cosaturated maps

We now have a toolset to answer the question!

CLASSES OF "ANODYNE" MORPHISMS

$$\text{InnHorn} := \{ \Delta_k^n \subset \Delta^n : 0 < k < n \}_{n \geq 2}$$

$$\text{LHorn} := \{ \Delta_k^n \subset \Delta^n : 0 \leq k < n \}_{n \geq 1}$$

$$\text{RHorn} := \{ \Delta_k^n \subset \Delta^n : 0 < k \leq n \}_{n \geq 1}$$

$$\text{Horn} := \{ \Delta_k^n \subset \Delta^n : 0 \leq k \leq n \}_{n \geq 1}$$

We will be interested in $\overline{\text{InnHorn}}$ since

Prop: If \mathcal{C} is an ∞ -category and $A \subseteq B$ is an inner anodyne inclusion, then any $f: A \rightarrow \mathcal{C}$ admits an extension

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{C} \\ \downarrow & & \nearrow \\ B & & \end{array}$$

Proof: The class $\left\{ X \hookrightarrow Y \mid \begin{array}{ccc} X & \xrightarrow{f} & \mathcal{C} \\ \downarrow & & \nearrow \\ Y & & \end{array} \text{ for all } \infty\text{-cat. } \mathcal{C} \text{ and } f: X \rightarrow \mathcal{C} \right\}$

is weakly saturated as we saw before.

Moreover, it contains the class InnHorn , therefore it must include $\overline{\text{InnHorn}}$. \square

$$I^n \subseteq \Delta^n$$

LIFTING CALCULUS & INNER FIBRATIONS

Move from "extensions" to "lifts"



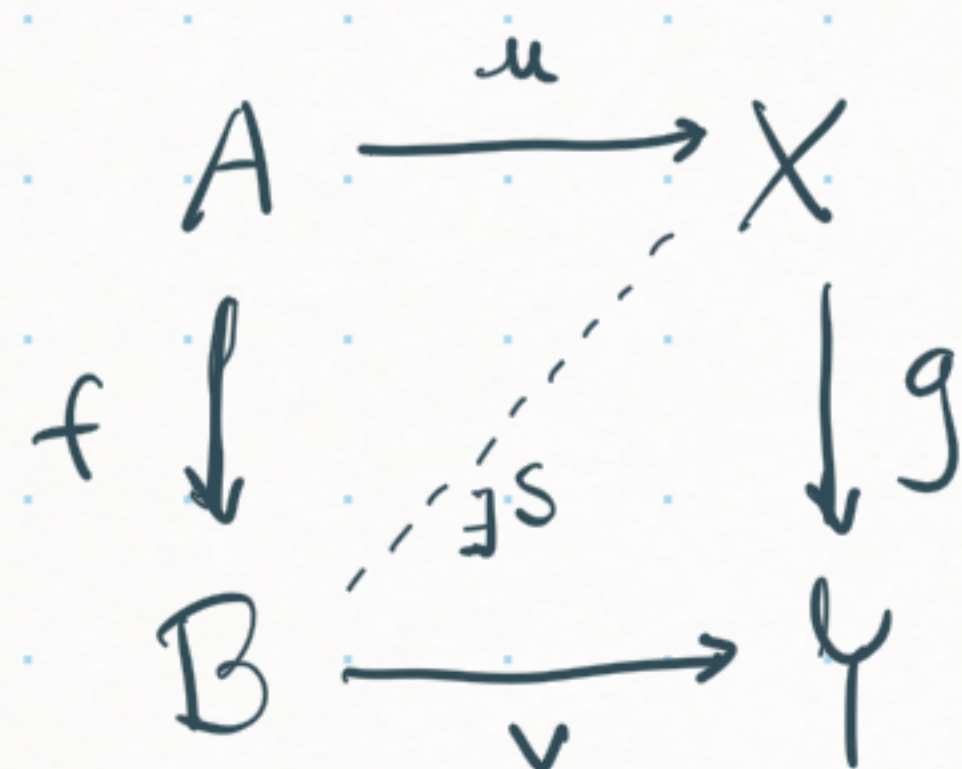
- Kan complex
- Weak Kan Complex

~>

- Kan Fibration
- Inner Fibration

Let $f: A \rightarrow B$ & $g: X \rightarrow Y$, a

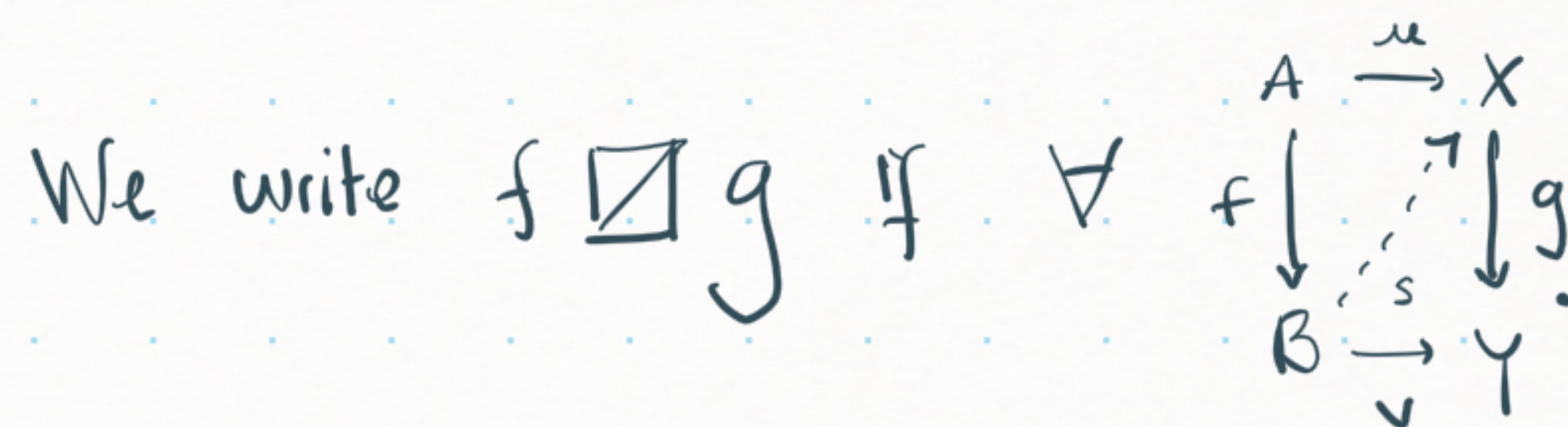
lifting problem for (f, g) is:



a pair $(u: A \rightarrow X, v: B \rightarrow Y)$

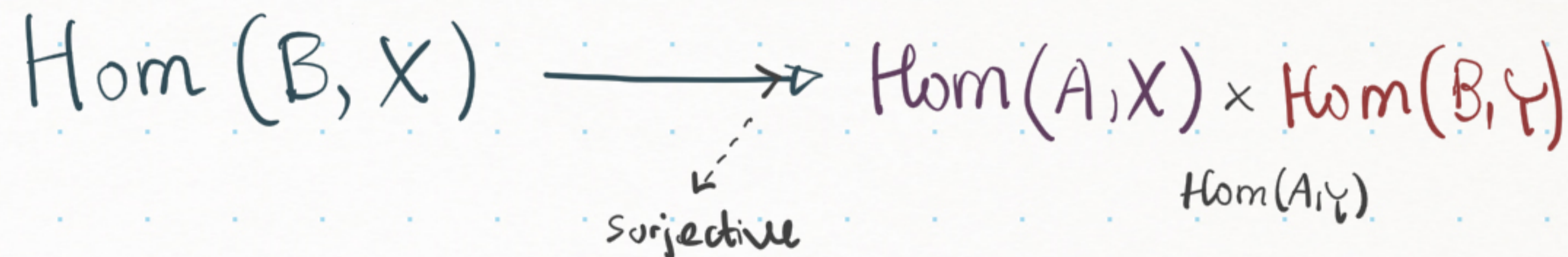
s.t. $g \circ s = v$
 $s \circ f = u$

• We call $s: B \rightarrow X$ a lift for the problem



Towards Enriched Lifting...

We can redefine $f \square g$ by:



$$S \longmapsto (g \circ s, s \circ f)$$

Surjectivity means that every lifting problem has a solution

Notation: $f \boxtimes g$ \rightsquigarrow f has the **left lifting property** w.r.t g
 \rightsquigarrow g has the **right lifting property** w.r.t f

$A \boxtimes B$ if any $f \in A, g \in B$ have $f \boxtimes g$

$A^\boxtimes := \{f \mid a \boxtimes f \ \forall a \in A\}$
 \rightsquigarrow right complement

$\boxtimes A := \{g \mid g \boxtimes a \ \forall a \in A\}$
 \rightsquigarrow left complement

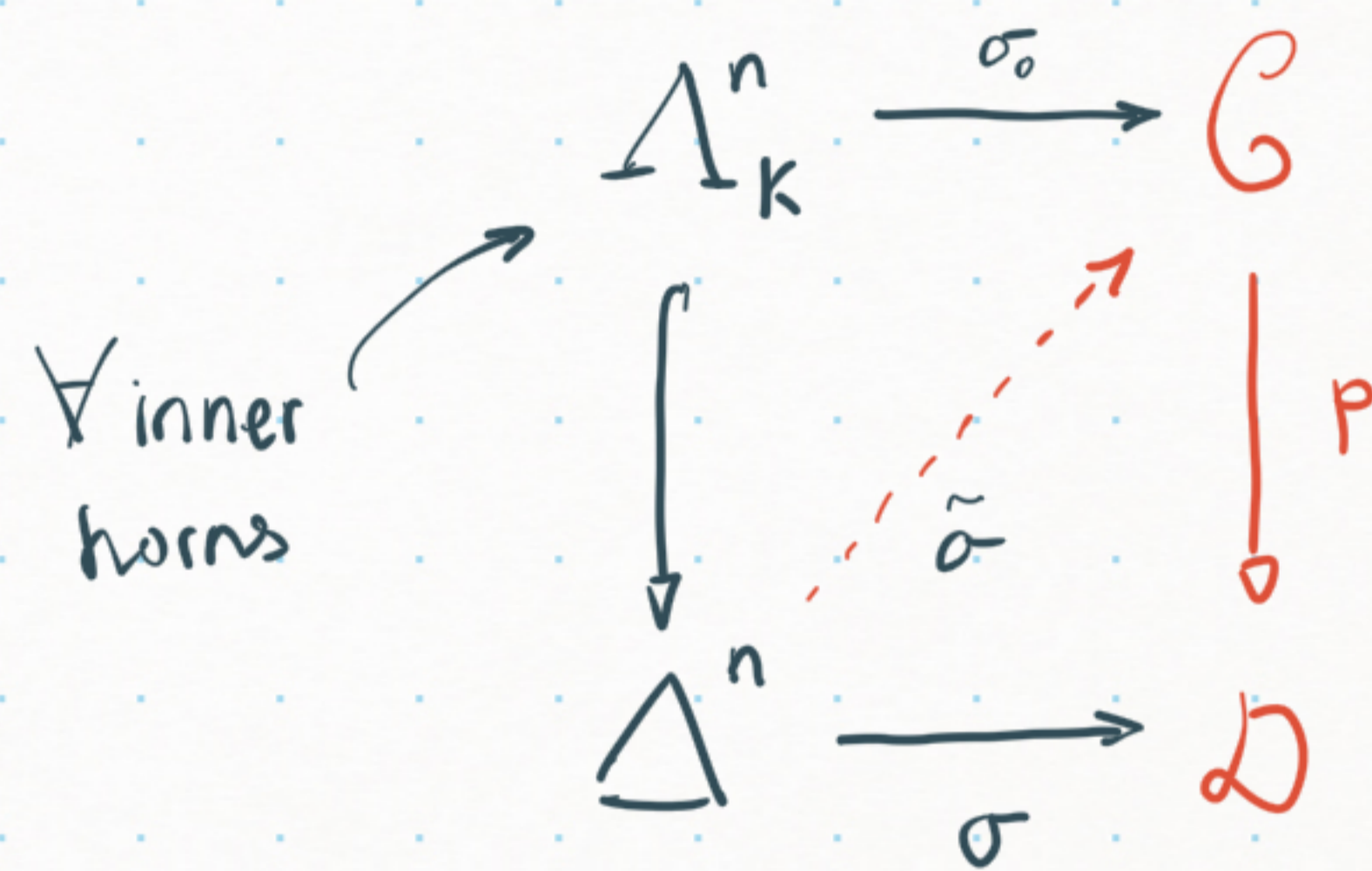
Prop: For any class A :

$\rightarrow \boxtimes A$ the **left complement** is a **weakly saturated class**

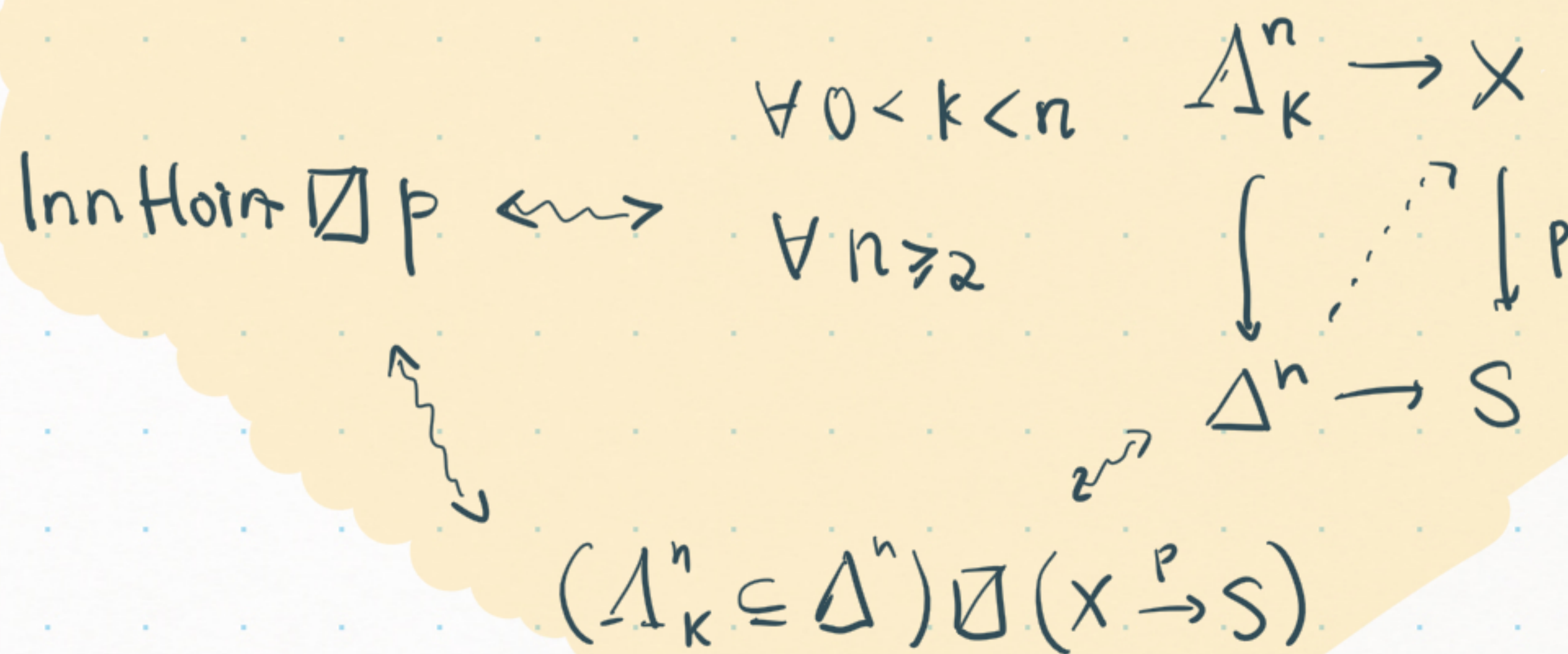
$\rightarrow A^\boxtimes$ the **right complement** is a **weakly cosaturated class**

\rightsquigarrow generalization of previous result!

INNER FIBRATIONS



Def: An **inner fibration** is a map $p: X \rightarrow S$ of simplicial sets such that:



DIAGRESSION ON INNER FIBRATIONS

Harder to motivate inner fibrations,
because they have no counterpart in
classical category theory!

$$f: X \rightarrow S$$

f is a trivial fibration



f is a Kan fibration



f is a left fibration



f is a right fibration



f is a CoCartesian fibration



f is a Cartesian fibration



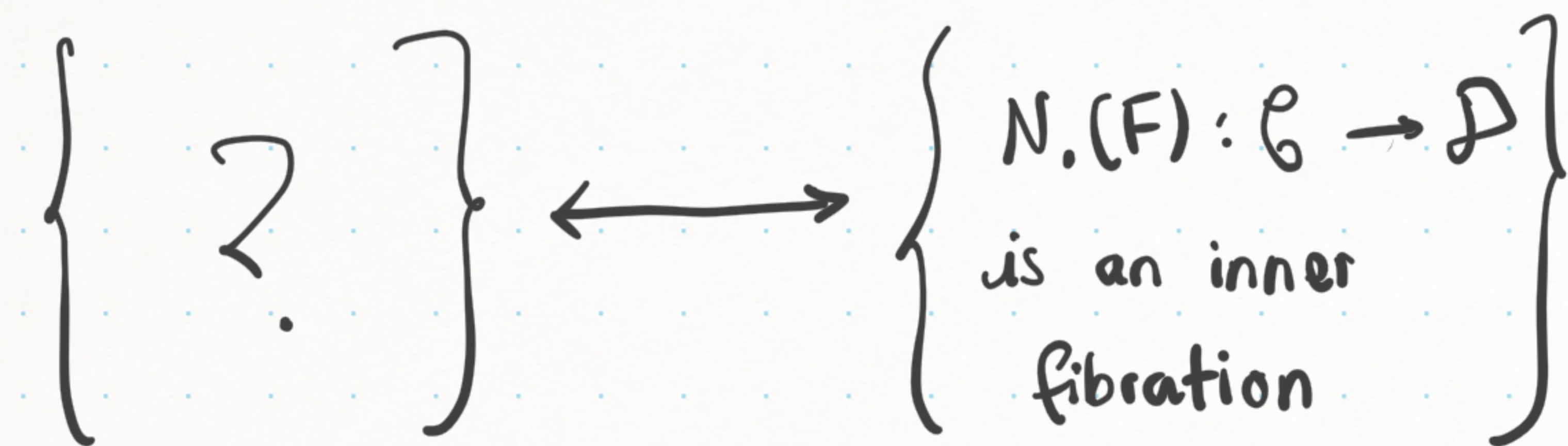
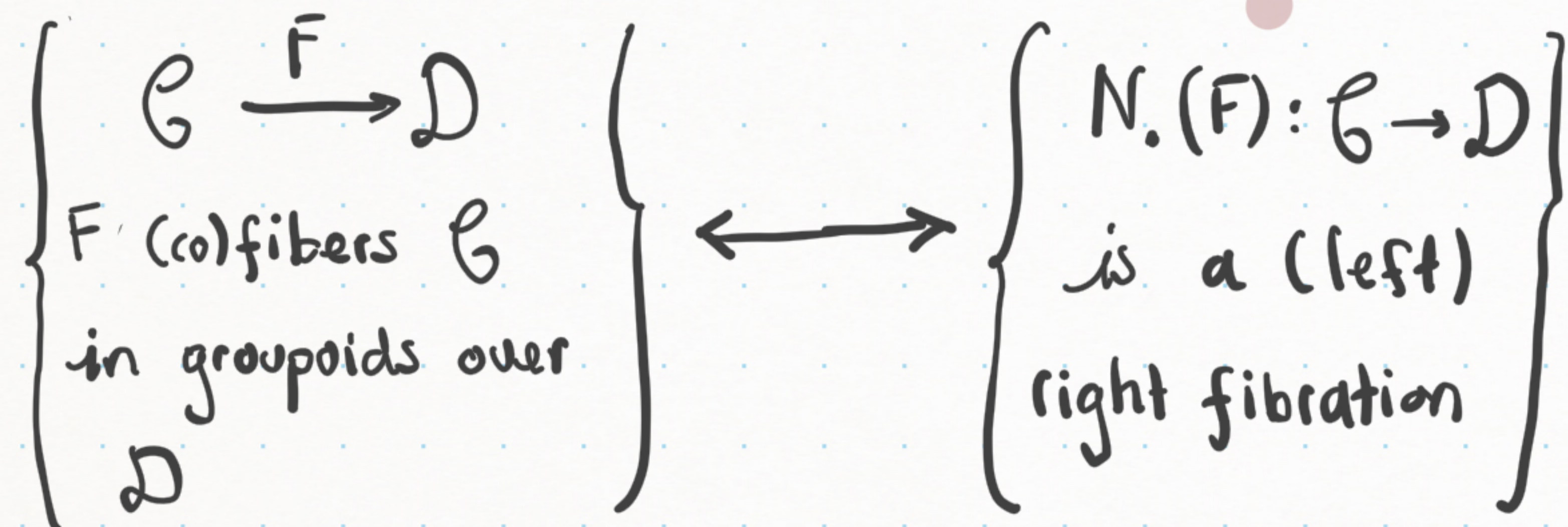
f is a Categorical
fibration



f is an inner
fibration

Very precise sense:

We can show $p^{-1}(s)$ are Kan Complex



$$\text{InnFib} := \text{InnHorn} \square$$



weakly cosaturated
(closed under base change, ...)

$$\rightsquigarrow \text{InnFib} = \overline{\text{InnFib}}$$

\square InnFib is weakly cosaturated



$$\square \text{InnFib} = \square (\text{InnHorn} \square) \cong \text{InnHorn}$$



$$\overline{\text{InnHorn}} = \square \text{InnFib}$$

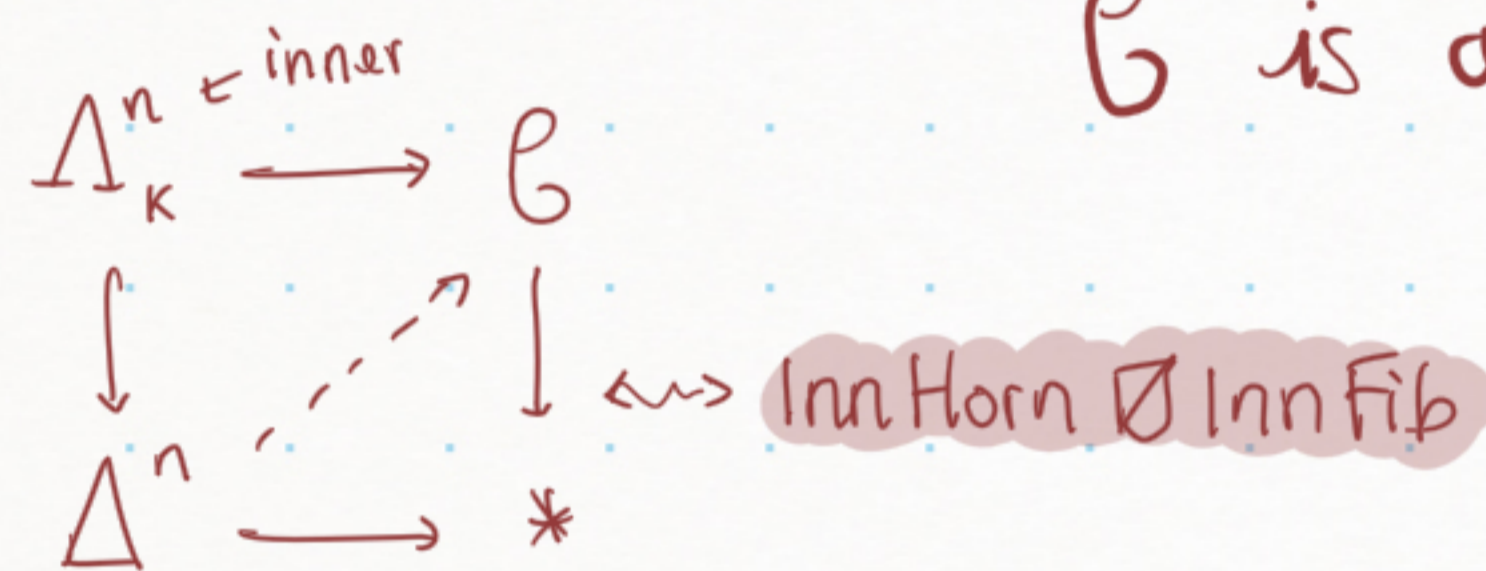
in other words:

$$\overline{\text{InnHorn}} \square \text{InnFib}$$

From this abstract reasoning:

- $\mathcal{C} \rightarrow *$ is an inner fibration iff

\mathcal{C} is an ∞ -category



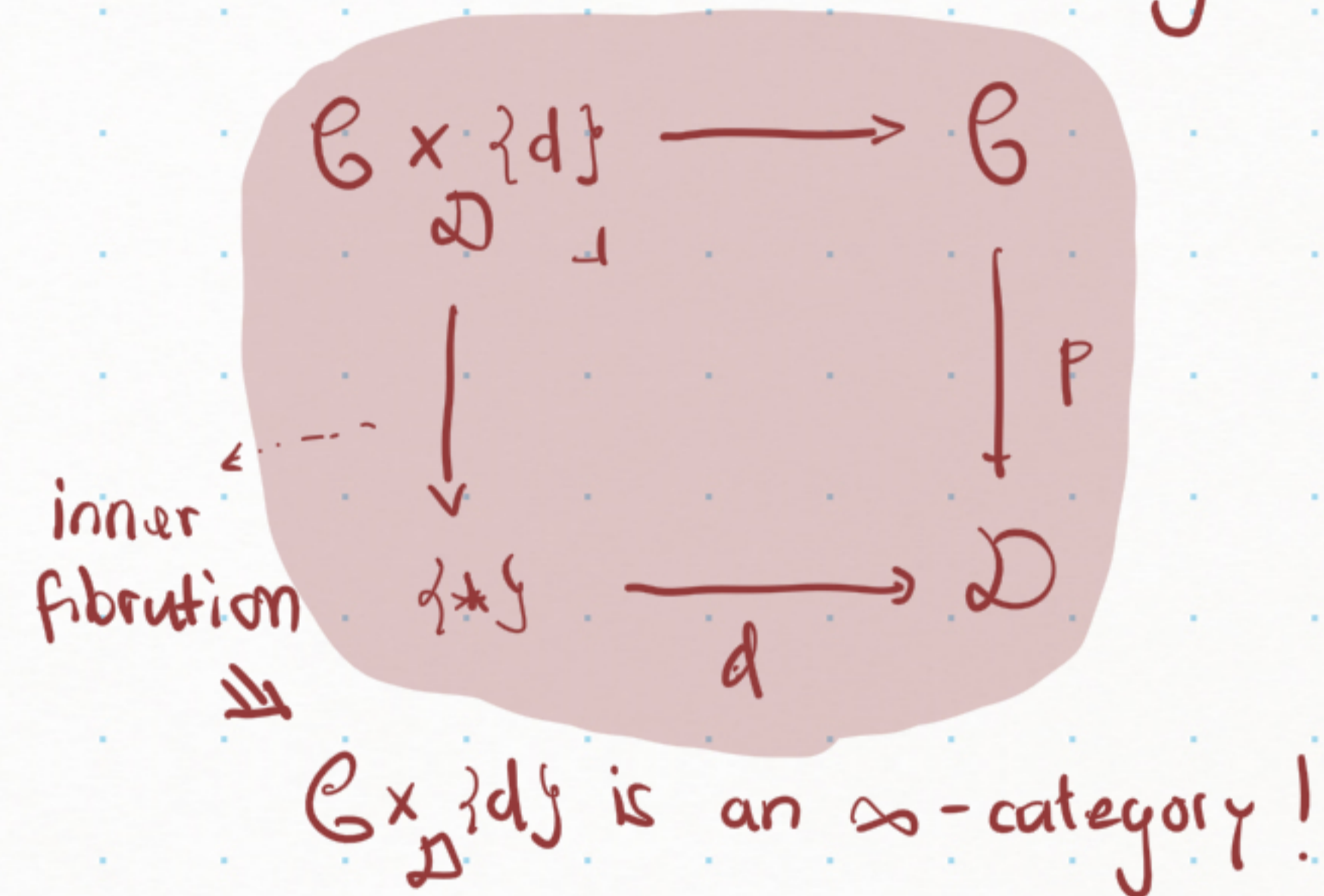
- If $p: \mathcal{C} \rightarrow \mathcal{D}$ is and p is an inner fibration and \mathcal{D} is an ∞ -category then \mathcal{C} is an ∞ -category

Why? InnFib is weakly saturated, in particular it's closed under compositions:

$$\mathcal{C} \xrightarrow{p} \mathcal{D} \rightarrow * \in \text{InnFib}$$

- The fibers of an inner fibration are ∞ -categories:

InnFib is closed under base change:



More Stuff (exercises): $\mathcal{C}' \hookrightarrow \mathcal{C}$

- $\mathcal{C}' \subseteq \mathcal{C}$ of a subcomplex is an inner fibration iff \mathcal{C}' is an ∞ -subcategory.
- $N_*(F): N_*\mathcal{C} \rightarrow N_*\mathcal{D}$ is an inner fibration for every functor $F: \mathcal{C} \rightarrow \mathcal{D}$.

