MAIN GOALS

Things we will prove today:

- $\mathrm{Fun}(X, \mathcal{C})$ is an ∞-category if $\mathcal{C}$ is an ∞-category.
- $\mathrm{Comp}_{\mathcal{C}}(f, g)$ is a contractible Kan complex iff $\mathcal{C}$ is an ∞-category.
- $\mathrm{Map}_{\mathcal{C}}(x, y)$ is an ∞-category (unsatisfactory, but we can't do more now...)

but we will prove many more things!

Outline

1. Introduce weakly saturated classes and inner-anodyne maps

2. Introduce :

   2.1 Lifting Calculus

   2.2 Inner Fibrations

3. Digression on Inner Fibrations

4. A tourist's view of :

   4.1 Small Object Argument & Weak Factorization

   4.2 Skeletal Filtrations

   4.3 Trivial Fibrations

5. Pushout-Product & Pullback-Hom (a.k.a enriched lifting)

6. Function Complexes of ∞-categories are ∞-categories

7. Sanity check: $\mathcal{C}^{\Delta^2} \longrightarrow \mathcal{C}^{\Lambda^2_1}$

   does what we want...

Weakly Saturated Classes and Inner-Anodyne Maps

- ∞-categories are defined by an "extension property"

- $\mathcal{C}$ is an ∞-Category if

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & \mathcal{C} \\ L \arrow[ur, dashed] & \end{tikzcd}$$

for every $K \subseteq L \in \{\Lambda^n_j \subseteq \Delta^n : n \geq 2, 0 < j < n\}$

Q: Is $\{\Lambda^n_j \subseteq \Delta^n : n \geq 2, 0<j<n\}$ precisely the class of morphisms with this extension property?

not really

Weakly Saturated Classes: A a class of morphisms

Such that:

1) Contains all isomorphisms $f : X \xrightarrow{\sim} Y$

2) is closed under cobase change

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] & Y \sqcup_X Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

3) is closed under composition

4) is closed under transfinite composition

5) is closed under coproducts

6) is closed under retracts

f is a retract of g

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

$X \to X' \to X = \mathrm{id}$, $Y \to Y' \to Y = \mathrm{id}$

Def: Given a class of maps S, we define its weak saturation to be the smallest saturated class containing S. $S \rightsquigarrow \overline{S}$

Examples: $\overline{\{\{0,1\} \to \{1\}\}} = \mathrm{Surj}$

$\overline{\{\emptyset \hookrightarrow \{1\}\}} = \mathrm{Mono}$

Really crucial result!

Prop: Consider a collection $\{C_\alpha\}$ of simplicial sets.

Let $\mathcal{A} := \left\{ i: A \to B \;\middle|\; \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \\ B \arrow[ur, dotted, "\exists g"'] & \end{tikzcd} \;\; \forall f \in \mathrm{Hom}(A, C_\alpha) \;\; \forall \alpha \right\}$

Then $\mathcal{A}$ is a weakly saturated class

Proof: We must show that this class

1) contains all isomorphisms: Assume that $A \xrightarrow{i} B$ is an isomorphism

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & C_\alpha \\ & B \arrow[ul, bend left, "i^{-1}"] \arrow[ur, dashed] & \end{tikzcd}$$

2) is closed under cobase change

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C_\alpha \\ B \arrow[ur, dotted, "\exists g"'] & \end{tikzcd} \quad \text{and let} \quad \begin{tikzcd} A \arrow[r, "p"] \arrow[d, "i"'] & C \arrow[d, "i'"] \\ B \arrow[r, "p'"'] & C \sqcup_A B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "i"'] & C \arrow[r, "h"] \arrow[d, "i'"] & C_\alpha \\ B \arrow[r, "p'"'] \arrow[urr, dashed] & C \sqcup_A B \arrow[ur, dashed] & \end{tikzcd}$$

$\vdots$

We could have defined a class of weakly cosaturated maps

We now have a toolset to answer the question!

CLASSES OF "ANODYNE" MORPHISMS

$$\mathrm{InnHorn} := \{\Lambda^n_k \subset \Delta^n : 0 < k < n\}_{n \geq 2}$$

$$\mathrm{LHorn} := \{\Lambda^n_k \subset \Delta^n : 0 \leq k < n\}_{n \geq 1}$$

$$\mathrm{RHorn} := \{\Lambda^n_k \subset \Delta^n : 0 < k \leq n\}_{n \geq 1}$$

$$\mathrm{Horn} := \{\Lambda^n_k \subset \Delta^n : 0 \leq k \leq n\}_{n \geq 1}$$

We will be interested in $\overline{\mathrm{InnHorn}}$ since

Prop: If $\mathcal{C}$ is an ∞-category and $A \subseteq B$ is an inner anodyne inclusion, then any $f : A \to \mathcal{C}$ admits an extension

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathcal{C} \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Proof: The class $\left\{ X \hookrightarrow Y \;\middle|\; \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ Y \arrow[ur, dashed] & \end{tikzcd} \text{ for all } \infty\text{-cat. } \mathcal{C} \text{ and } f : X \to \mathcal{C} \right\}$

is weakly saturated as we saw before. Moreover, it contains the class InnHorn, therefore it must include $\overline{\mathrm{InnHorn}}$. $\square$

$I^n \subseteq \Delta^n$

LIFTING CALCULUS & INNER FIBRATIONS

Move from "extensions" to "lifts"

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B \arrow[ur, dashed] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

- Kan complex
- Weak Kan complex

$\rightsquigarrow$

- Kan Fibration
- Inner Fibration

Let $f : A \to B$ & $g : X \to Y$, a <u>lifting problem</u> for (f, g) is:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\exists s"] & Y \end{tikzcd}$$

a pair $(u : A \to X, v : B \to Y)$

s.t $g \circ s = v$

$s \circ f = u$

- We call $s : B \to X$ a <u>lift</u> for the problem

We write $f \boxslash g$ if $\forall$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

Towards Enriched Lifting ...

We can redefine $f \boxslash g$ by :

$$\mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)$$

Surjective

$$s \longmapsto (g \circ s \,,\, s \circ f)$$

Surjectivity means that every lifting problem has a solution

Notation: $f \boxslash g$ ⇝ f has the left lifting property w.r.t g

⇝ g has the right lifting property w.r.t f

$\mathcal{A} \boxslash \mathcal{B}$ if every $f \in \mathcal{A}$, $g \in \mathcal{B}$ have $f \boxslash g$

$\mathcal{A}^{\boxslash} := \{ f \mid a \boxslash f \ \forall a \in \mathcal{A} \}$

⇝ right complement

${}^{\boxslash}\mathcal{A} := \{ g \mid g \boxslash a \ \forall a \in \mathcal{A} \}$

⇝ left complement

Prop: For any class $\mathcal{A}$:

→ ${}^{\boxslash}\mathcal{A}$ the left complement is a weakly saturated class

→ $\mathcal{A}^{\boxslash}$ the right complement is a weakly cosaturated class

generalization of previous result!

INNER FIBRATIONS

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dashed, "\tilde{\sigma}"] & \mathcal{D} \end{tikzcd}$$

$\forall$ inner horns

Def: An inner fibration is a map $p : X \to S$ of simplicial sets such that:

$$\text{InnHorn} \boxslash p \iff \forall\, 0 < k < n, \ \forall\, n \geq 2 \quad \begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

$$\Updownarrow$$

$$(\Lambda^n_k \subseteq \Delta^n) \boxslash (X \xrightarrow{p} S)$$

Digression on Inner Fibrations

Harder to motivate inner fibrations, because they have no counterpart in classical category theory!

$f : X \to S$

$$\begin{tikzcd}[column sep=small]
 & f \text{ is a trivial fibration} \arrow[d, Rightarrow] & \\
 & f \text{ is a Kan fibration} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
f \text{ is a left fibration} \arrow[d, Rightarrow] & & f \text{ is a right fibration} \arrow[d, Rightarrow] \\
f \text{ is a CoCartesian fibration} \arrow[dr, Rightarrow] & & f \text{ is a Cartesian fibration} \arrow[dl, Rightarrow] \\
 & f \text{ is a Categorical fibration} \arrow[r, Rightarrow] & f \text{ is an inner fibration}
\end{tikzcd}$$

Very precise sense:

We can show $p^{-1}(s)$ are Kan Complex

$$\left\{\begin{array}{l} \mathcal{C} \xrightarrow{F} \mathcal{D} \\ F \text{ (co)fibers } \mathcal{C} \\ \text{in groupoids over} \\ \mathcal{D} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} N_\bullet(F) : \mathcal{C} \to \mathcal{D} \\ \text{is a (left)} \\ \text{right fibration} \end{array}\right\}$$

$$\left\{\ ?\ \right\} \longleftrightarrow \left\{\begin{array}{l} N_\bullet(F) : \mathcal{C} \to \mathcal{D} \\ \text{is an inner} \\ \text{fibration} \end{array}\right\}$$

$\mathrm{InnFib} := \mathrm{InnHorn}^{\boxslash}$

weakly cosaturated (closed under base change, ...) $\rightsquigarrow \mathrm{InnFib} = \overline{\mathrm{InnFib}}$

${}^{\boxslash}\mathrm{InnFib}$ is weakly cosaturated

$\Downarrow$

${}^{\boxslash}\mathrm{InnFib} = {}^{\boxslash}(\mathrm{InnHorn}^{\boxslash}) \supseteq \mathrm{InnHorn}$

$\Downarrow$

$\overline{\mathrm{InnHorn}} = {}^{\boxslash}\mathrm{InnFib}$

in other words:

$$\overline{\mathrm{InnHorn}} \boxslash \mathrm{InnFib}$$

From this abstract reasoning:

- $\mathcal{C} \to *$ is an inner fibration iff $\mathcal{C}$ is an ∞-category

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

Λ^n_k inner; $\rightsquigarrow$ InnHorn $\pitchfork$ InnFib

- If $p: \mathcal{C} \to \mathcal{D}$ is and p is and inner fibration and $\mathcal{D}$ is an ∞-category then $\mathcal{C}$ is an ∞-category

Why? InnFib is weakly cosaturated, in particular it's closed under compositions:

$$\mathcal{C} \xrightarrow{p} \mathcal{D} \to * \in \text{InnFib}$$

- The fibers of an inner fibration are ∞-categories:

InnFib is closed under base change:

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \{d\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "p"] \\ \{*\} \arrow[r, "d"] & \mathcal{D} \end{tikzcd}$$

inner fibration $\Rightarrow$ $\mathcal{C} \times_{\mathcal{D}} \{d\}$ is an ∞-category!

More Stuff (exercises): $\mathcal{C}' \xhookrightarrow{i} \mathcal{C}$

- $\mathcal{C}' \subseteq \mathcal{C}$ of a subcomplex is an inner fibration iff $\mathcal{C}'$ is an ∞-subcategory.
- $N_\bullet(F): N_\bullet\mathcal{C} \to N_\bullet\mathcal{D}$ is an inner fibration for every functor $F: \mathcal{C} \to \mathcal{D}$.