

WEAK FACTORIZATION

SYSTEMS

Fact: Any map of simplicial sets f ,

can be factored as

$$f = p \circ j$$

where $p \in \text{InnFib} \rightsquigarrow$ inner fibration

$j \in \overline{\text{InnHorn}} \rightsquigarrow$ inner anodyne

This follows from:

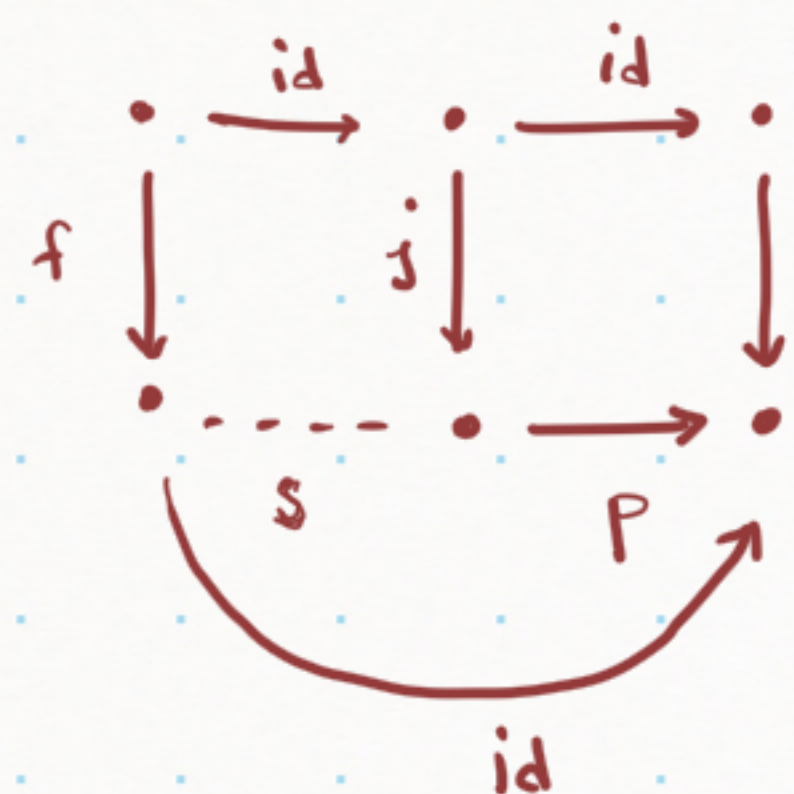
Prop: For any A collection of maps in $s\text{Set}$,

$$\bar{S} = \square(S^\square)$$

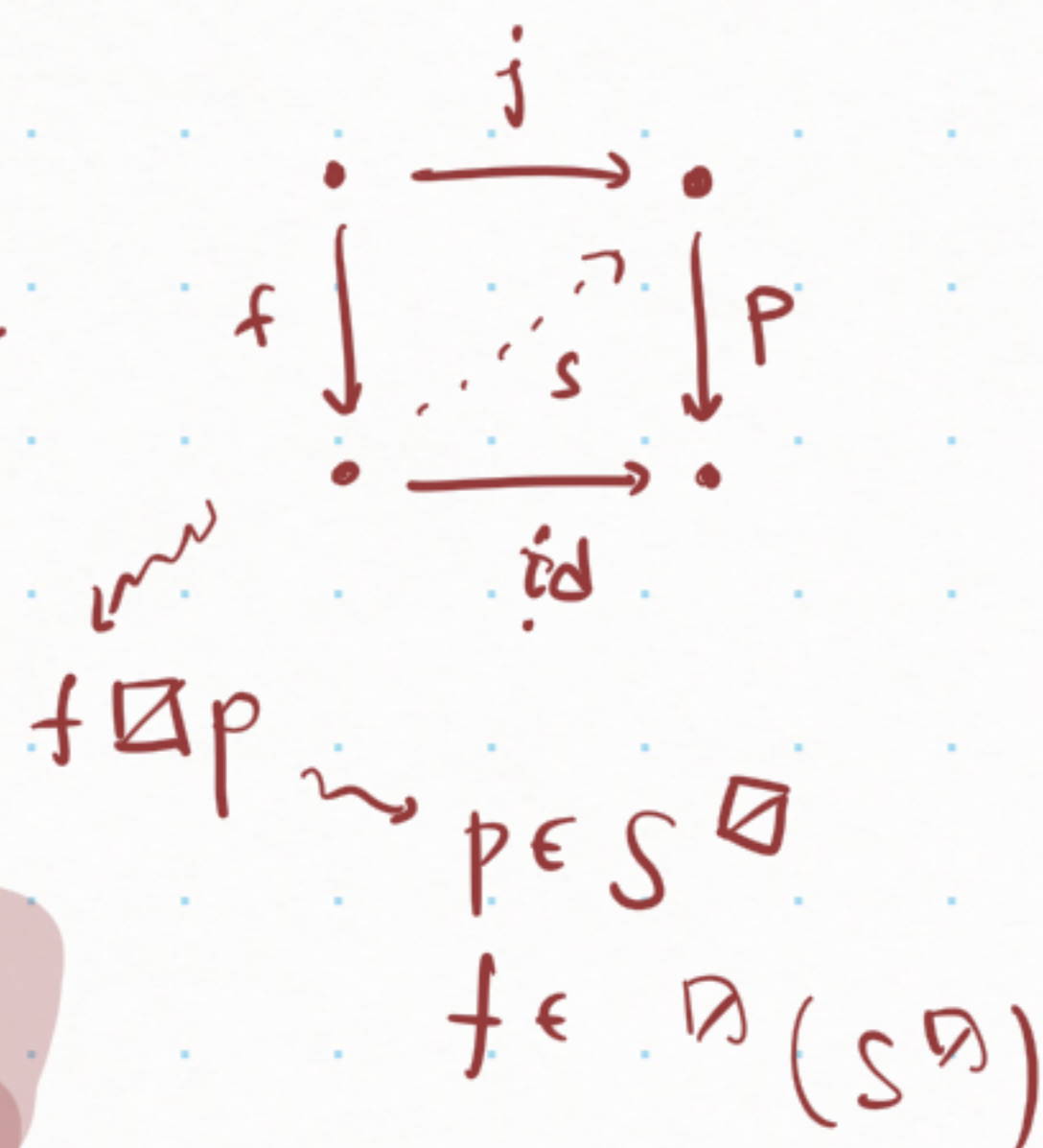
Proof: $\bar{S} \subseteq \square(S^\square)$ from the fact that $\square(-)$ is a weakly saturated class and $S \subseteq \square(S^\square)$.

$\bar{S} \supseteq \square(S^\square)$: Consider $f \in S^\square$,

the small object argument tells us we can choose $j \in \bar{S}$, $p \in S^\square$ s.t. $f = p \circ j$



S exists because



This wizardry has a name: "retract trick"

In our case $S = \text{InnHorn}$ $\rightsquigarrow \bar{S} = \text{inner-anodyne}$
 $\rightsquigarrow S \square = \text{Inner fibrations}$

$$\overline{\text{InnHorn}} = \square \text{InnFib}$$

&

$$\overline{\text{InnHorn}} \square = \text{InnFib}$$

The pair $(\overline{\text{InnHorn}}, \text{InnFib})$ is a particular case of a **weak factorization system**.

$$\partial \Delta^n = \bigcup_{\{ij\} \in [n]} \Delta^{[n] \setminus \{ij\}}$$

$$\text{Cell} := \{ \partial \Delta^n \subseteq \Delta^n \}_{n \geq 0}$$

$$(\overline{\text{Cell}}, \text{Cell}^{\square})$$

$$\text{TrivFib} := \text{Cell}^{\square}$$

$$A \hookrightarrow X$$

$$X = \text{colim}_K AU(SK_K(X))$$

$$AU(SK_0(X)) \hookrightarrow AU(SK_1(X)) \hookrightarrow \dots$$

$$\overline{\text{Cell}} \subseteq \text{Mono}$$

$$\boxed{\overline{\text{Cell}} = \text{Mono}}$$

$$A \hookrightarrow X \underset{\text{iso.}}{\sim} A \subseteq X$$

$$\coprod_{a \in X_K^{\text{nd}} \mid A_K^{\text{nd}}} \partial \Delta^K \longrightarrow AU(SK_{K-1}(X))$$



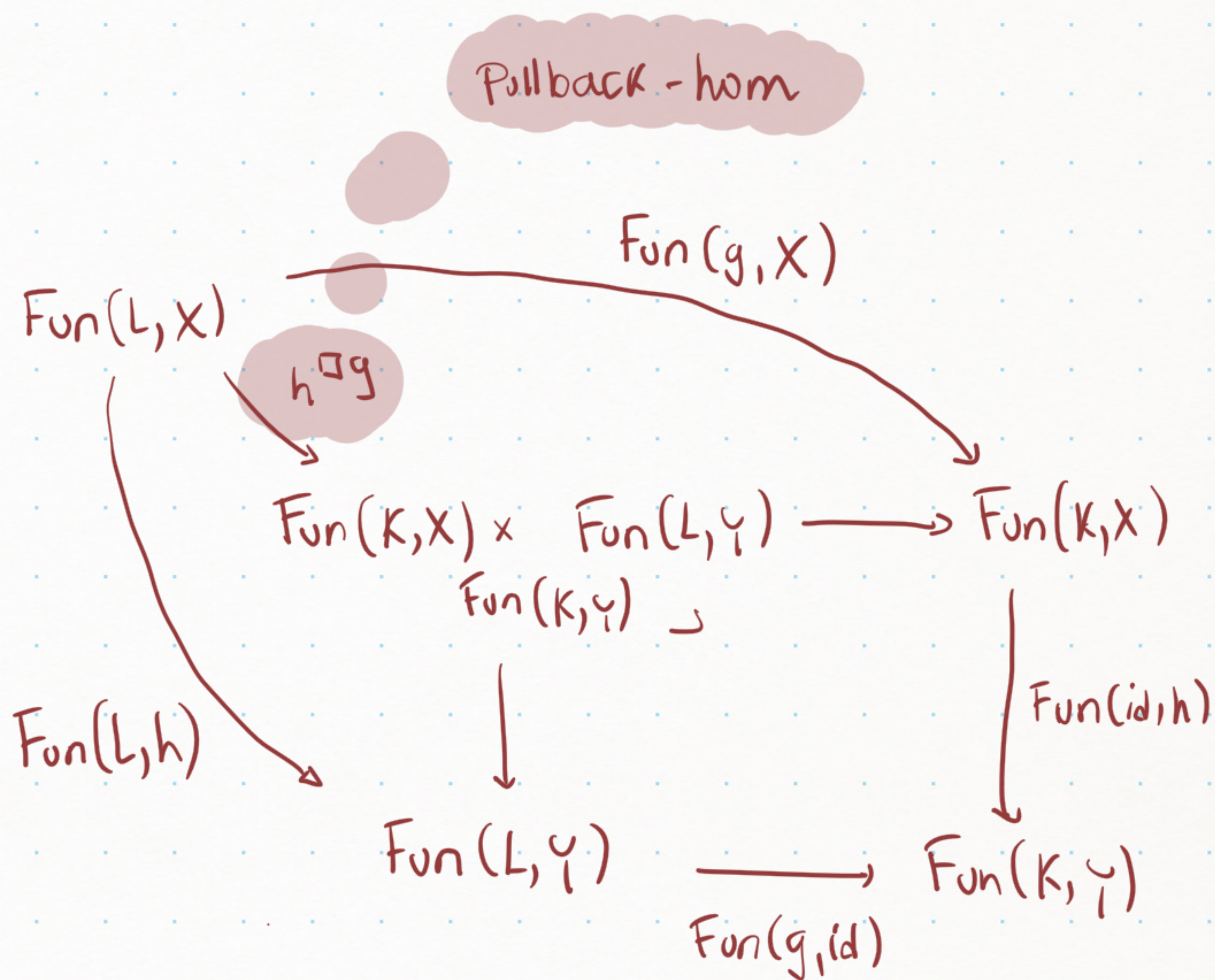
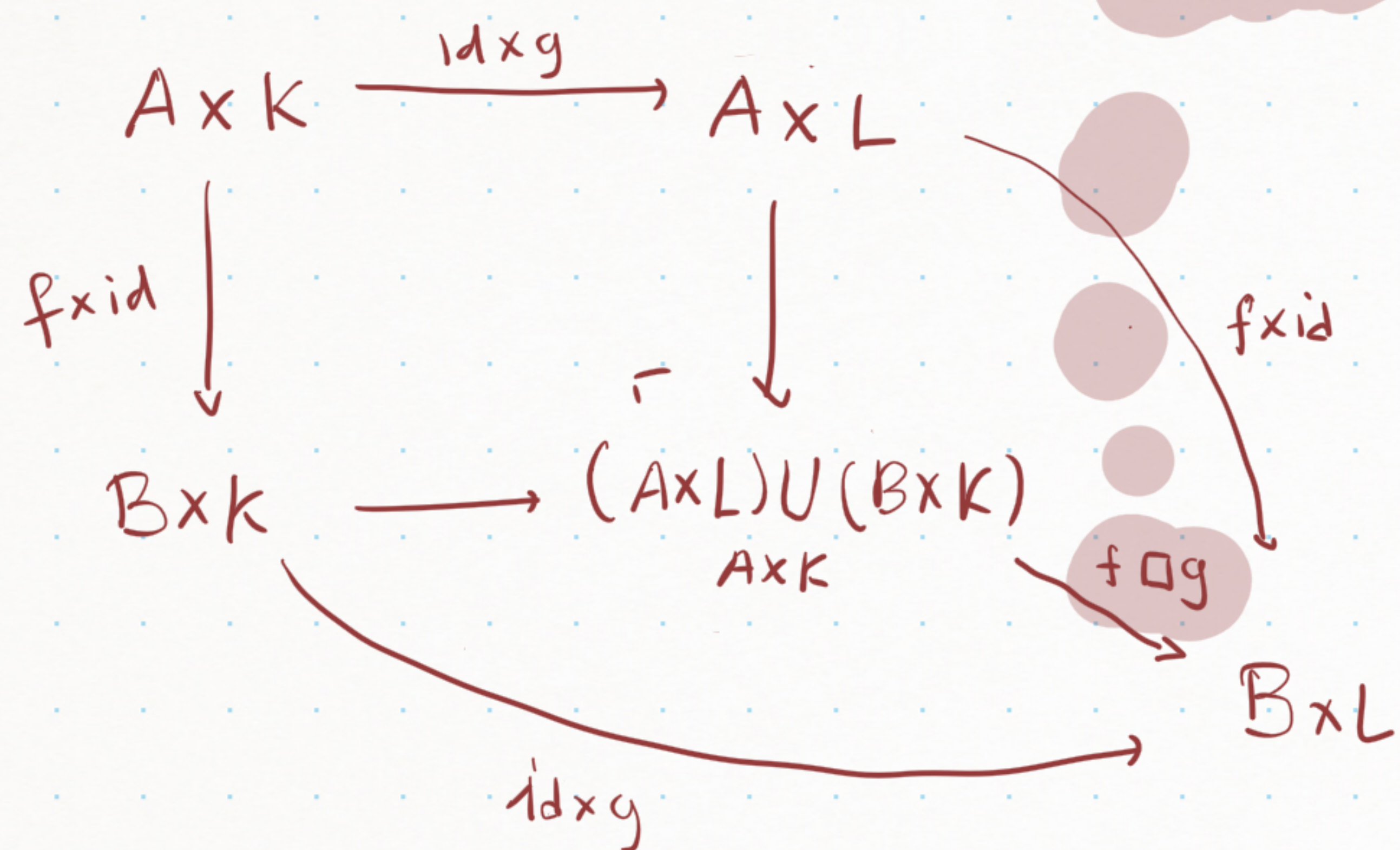
$$\coprod_{a \in X_K^{\text{nd}} \mid A_K^{\text{nd}}} \Delta^K$$

$$\begin{array}{c} \longrightarrow AU(SK_{K-1}(X)) \\ \downarrow \\ \longrightarrow AU(SK_K(X)) \end{array}$$

PUSHOUT-PRODUCT & PULLBACK-HOM

$f: A \rightarrow B$
 $g: K \rightarrow L$
 $h: X \rightarrow Y$

maps of
Simplicial Sets



Remarks:

- $h \square g$ \rightsquigarrow enriched lifting $(h \square g)_0$ is the usual lifting set up!
- $\square: Ar(sSet) \times Ar(sSet) \rightarrow Ar(sSet)$ defines a symmetric monoidal structure & $- \square g^{-1} (-) \square g$

ADJUNCTION OF LIFTING PROBLEMS

Prop: $(f \circ g) \dashv h$ iff $f \dashv (h \circ g)$

$$\begin{array}{ccc}
 (A \times L) \cup (B \times K) \xrightarrow{h} X & & A \xrightarrow{\tilde{u}} \text{Fun}(L, X) \\
 \begin{array}{ccc}
 f \circ g \downarrow & \nearrow s & \downarrow h \\
 B \times L & \xrightarrow{\omega} & Y
 \end{array} & \Leftrightarrow & \begin{array}{ccc}
 f \downarrow & \nearrow \tilde{s} & \downarrow h \circ g \\
 B & \xrightarrow{(\tilde{v}, \tilde{w})} & \text{Fun}(K, X) \times \text{Fun}(L, Y) \\
 & & \text{Fun}(B, Y)
 \end{array}
 \end{array}$$

Proof uses the adjunction $- \circ g \dashv (-) \circ g$, but it's checkable with some thought...

Note: If $A = \emptyset$, $K = \emptyset$ or $Y = *$ hold, then:
 for instance if $K = \emptyset$, $Y = *$

$$(A \times L \xrightarrow{f \times L} B \times L) \square (X \rightarrow *) \text{ iff}$$

$$(A \xrightarrow{f} B) \square (\text{Fun}(L, X) \rightarrow *)$$

Prop: S, T any collections of maps

$$\overline{S \square T} \subseteq \overline{S \square T}$$

↓
+

Lemma: $\text{InnHorn} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$

$(\Delta^n_k \in \Delta^n) \square (\partial \Delta^m \in \Delta^m)$ is inner-analytic

→ $\overline{\text{InnHorn} \square \text{Cell}} \subseteq \overline{\text{InnHorn}}$

Prop: 1) If $i: A \rightarrow B$ is inner-anodyne and

$j: K \hookrightarrow L$ is a monomorphism

$$i \sqcup_j: (A \times L) \cup_{A \times K} (B \times K) \rightarrow B \times L$$

is inner-anodyne

2) If $j: K \hookrightarrow L$ is a monomorphism

