Last Time:

- We defined weakly saturated classes
- We introduced trivial fibrations and inner fibrations

last things we proved:

$$\overline{\mathrm{Cell}} = \mathrm{Mono} \rightsquigarrow$$ "the weak saturation of the $\{\partial\Delta^n \hookrightarrow \Delta^n\}$ are the monomorphisms."

$$sk_0(X) \hookrightarrow sk_1(X) \hookrightarrow \cdots$$

Lifting Adjunction

$$
\begin{tikzcd}
A\times L \coprod_{A\times K} B\times K \arrow[r, "{(u,v)}"] \arrow[d, "f\square g"'] & X \arrow[d, "h"] \\
B\times L \arrow[r, "w"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\tilde{u}"] \arrow[d, "f"'] & \mathrm{Fun}(L,X) \arrow[d, "h^{\square g}"] \\
B \arrow[r, "{(\tilde{w},\tilde{v})}"'] \arrow[ur, dashed, "\tilde{s}"] & \mathrm{Fun}(L,Y) \times_{\mathrm{Fun}(B,Y)} \mathrm{Fun}(K,X)
\end{tikzcd}
$$

$$- \square g \dashv (-)^{\square g}$$

Special Cases of Interest:

$A = \emptyset$

$K = \emptyset$

$Y = *$

$K = \emptyset, \ Y = *$

$A \xrightarrow{f} B$

$\emptyset \xhookrightarrow{g} L$

$X \xrightarrow{h} *$

$$\begin{tikzcd} A \times L \arrow[r] \arrow[d] & X \arrow[d] \\ B \times L \arrow[r] \arrow[ur, dashed, "s"] & * \end{tikzcd} \iff \begin{tikzcd} A \arrow[r] \arrow[d] & \underline{\mathrm{Fun}(L,X)} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "\tilde{s}"] & * \end{tikzcd}$$

∞-category

$C \longrightarrow * \rightsquigarrow$ inner fibration

End Goal:

Prove the following Proposition

① If $i : A \to B$ is inner-anodyne and $j : K \hookrightarrow L$ is a monomorphism, then

$$i \,\square\, j : (A \times L) \underset{A \times K}{\cup} (B \times K) \longrightarrow B \times L$$

is inner-anodyne

② If $j : K \hookrightarrow L$ is a monomorphism and $p : X \to Y$ is an inner fibration,

then

$$p^{\square j} : \mathrm{Fun}(L,X) \longrightarrow \mathrm{Fun}(K,X) \underset{\mathrm{Fun}(K,Y)}{\times} \mathrm{Fun}(L,Y)$$

is an inner fibration

(3) If $i : A \to B$ is inner-anodyne and $p : X \to Y$ is an inner fibration

then $$p^{\square i} : \mathrm{Fun}(B, X) \longrightarrow \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A, Y)} \mathrm{Fun}(B, Y)$$

is a trivial fibration.

$$\overline{\mathrm{Cell}} = \mathrm{Mono}$$

$$\mathrm{TrivFib} := \overline{\mathrm{Cell}}^{\boxslash}$$

These statements can be summarized:

- $\overline{\mathrm{InnHorn}} \,\square\, \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$
- $\mathrm{InnFib}^{\square \overline{\mathrm{Cell}}} \subseteq \mathrm{InnFib}$
- $\mathrm{InnFib}^{\square \overline{\mathrm{InnHorn}}} \subseteq \mathrm{TrivFib}$

$\boxslash$

What can we do with this :

⇝ If $i : A \to B$ is inner anodyne, then so is $i \times id_L : A \times L \to B \times L$

$$i \times id_L = i \,\square\, (\emptyset \hookrightarrow L)$$

⇝ If $p : X \to Y$ is an inner fibration, then so is $\mathrm{Fun}(L, p) : \mathrm{Fun}(L, X) \to \mathrm{Fun}(L, Y)$

$$(X \xrightarrow{p} Y)^{\square(\emptyset \subseteq L)} \nearrow$$

⇝ if $j : K \hookrightarrow L$ a mono, $\mathcal{C}$ an ∞-category $\quad \mathcal{C} \to *$

$$(\mathcal{C} \to *)^{\square(K \hookrightarrow L)} \rightsquigarrow \mathrm{Fun}(j, \mathcal{C}) : \mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C}) \text{ is an inner fibration}$$

⇝ if $i : A \to B$ is inner anodyne, $\mathcal{C}$ an ∞-category

$$\mathrm{Fun}(i, \mathcal{C}) : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \text{ trivial fibration}$$

As a special case:

Thm: For $\mathcal{C}$ an ∞-category and L a simplicial set, $\mathrm{Fun}(L, \mathcal{C})$ is an ∞-category.

mono

inner-anodyne

Proof: $$(\Lambda^n_j \subset \Delta^n) \,\square\, (\phi \subseteq K) = (\Lambda^n_j \times K \longrightarrow \Delta^n \times K)$$

Inner anodyne

$0 < j < n$

$\forall n \geq 2$

$$\begin{tikzcd}
\Lambda^n_j \times K \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d] \\
\Delta^n \times K \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}
\quad \Leftarrow \quad
\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d, hook] & \mathrm{Fun}(K, \mathcal{C}) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

$\Rightarrow \mathrm{Fun}(K, \mathcal{C})$ is an ∞-category

HOW TO PROVE THE THEOREM?

Lemma 1: Let S and T be two classes of maps, then

$$\boxed{\bar{S} \,\square\, T \subseteq \bar{S} \,\square\, \bar{T}} \subseteq \overline{S \,\square\, T}$$

Lemma 2: For $0 < j < n$, the inclusion $\Lambda^n_j \hookrightarrow \Delta^n$ is a retract of

$$\Lambda^n_j \times \Delta^2 \coprod_{\Lambda^n_j \times \Lambda^2_1} \Delta^n \times \Lambda^2_1 \longrightarrow \Delta^n \times \Delta^2$$

Lemma 3: All of the following classes of maps generate the set of inner-anodyne maps:

- $S_1 = \{\Lambda^n_j \hookrightarrow \Delta^n\}_{0<j<n}$;
- $S_2 = \{(K \hookrightarrow L) \square (\Lambda^2_1 \subseteq \Delta^2)\}$ for all monomorphisms $K \hookrightarrow L$
- $S_3 = \{(\partial\Delta^n \hookrightarrow \Delta^n) \square (\Lambda^2_1 \hookrightarrow \Delta^2)\}$ for all $n \geq 0$
- $S_4 = \{(K \hookrightarrow L) \square (\Lambda^n_j \hookrightarrow \Delta^n)\}$ for all mono $K \hookrightarrow L$ and all inner-horns.

Lemma 1: $\bar{S} \square T \subseteq \bar{S} \square \bar{T} \subseteq \overline{S \square T}$

Proof: (Small Obj argument + lifting adjunction) $\bar{S} \square T \subseteq \bar{S} \square \bar{T}$ ✓

$\bar{S} \square \bar{T} \subseteq \overline{S \square T}$

· $\mathcal{F} = (S \square T)^{\boxslash}$ per the small object argument ${}^{\boxslash}\mathcal{F} = \overline{(S \square T)}$

$(\bar{S} \square \bar{T}) \boxslash \mathcal{F}$

$(\bar{S} \square T) \boxslash \mathcal{F}$

$(\bar{S} \square \bar{T}) \boxslash \mathcal{F}$

$\mathcal{A} = \{ a \mid (a \square T) \boxslash \mathcal{F} \}$

adjunction $= \{ a \mid a \boxslash (\mathcal{F}^{\square T}) \} \rightsquigarrow \mathcal{A} = {}^{\boxslash}(\mathcal{F}^{\square T}) \Rightarrow \mathcal{A}$ is weakly saturated and it contains S, hence $\bar{S} \subseteq \mathcal{A}$

$$\bar{S} \subseteq \mathcal{A} \rightsquigarrow (\bar{S} \,\square\, T) \boxslash \tilde{\mathcal{F}}$$

$$\mathcal{B} = \{ b \mid (\bar{S} \,\square\, b) \boxslash \mathcal{F} \} \rightsquigarrow (\bar{S} \,\square\, \bar{T}) \boxslash \mathcal{F} \Rightarrow \bar{S} \,\square\, \bar{T} \subseteq \overline{S \,\square\, T}. \quad \square$$

<u>Lemma</u>: $0 < j < n$, the inclusion $\Lambda^n_j \hookrightarrow \Delta^n$ is a retract of

$$\underset{\Lambda^n_j \times \Lambda^2_1}{\Lambda^n_j \times \Delta^2 \bigcup \Delta^n \times \Lambda^2_1} \longrightarrow \Delta^n \times \Delta^2$$

<u>Proof</u>:

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & \Lambda^n_j \times \Delta^2 \cup \Delta^n \times \Lambda^2_1 \arrow[r] \arrow[d, hook] & \Lambda^n_j = \mathrm{id} \arrow[d] \\
\Delta^n \arrow[r, "s"'] & \Delta^n \times \Delta^2 \arrow[r, "r"'] & \Delta^n = \mathrm{id}
\end{tikzcd}$$

$$[n] \xrightarrow{s} [n] \times [2] \xrightarrow{r} [n]$$

$$i \longmapsto \begin{cases} (i,0) & \text{if } i<j \\ (i,1) & \text{if } i=j \\ (i,2) & \text{if } i>j \end{cases}$$

$$(i,k) \longmapsto \begin{cases} i & \text{if } i<j \text{ and } k=0 \\ i & \text{if } i>j \text{ and } k=2 \\ j & \text{else} \end{cases}$$

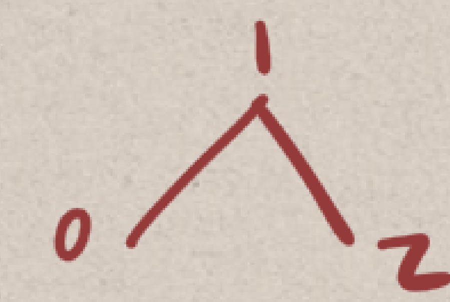
① $rs = id$ $\quad s : \Delta^n \to \Delta^n \times \Delta^2$

② $s(\Lambda^n_j) \subseteq \Lambda^n_j \times \Delta^2 \cup \Delta^n \times \Lambda^2_1$

③ $r(\Lambda^n_j \times \Delta^2 \cup \Delta^n \times \Lambda^2_1) \subseteq \Lambda^n_j$

$s(\Lambda^n_j) \subseteq \Lambda^n_j \times \Delta^2$

$$\begin{tikzcd} {[n]} \arrow[r, "s"] \arrow[dr, "id"'] & {[n]\times[2]} \arrow[d] \\ & {[n]} \end{tikzcd}$$



3 a $r(\Lambda^n_j \times \Delta^2) \subseteq \Lambda^n_j$

$f : [k] \to [n]$ s.t. $\exists m \in [n]\setminus\{j\}$ $im(f) \not\ni m$

$\alpha : [k] \to [2]$

$$[k] \xrightarrow{(f,\alpha)} [n]\times[2] \xrightarrow{r} [n]$$

$$(i,k) \longmapsto i \text{ or } j$$

$im(r\circ(f,\alpha)) \not\ni m$

hence represents a k-simplex of Λ^n_j

b $r(\Delta^n \times \Lambda^2_1) \subseteq \Lambda^n_j$

$\beta : [k] \to [n]$

$f : [k] \to [2]$ $im(f) \not\ni 0,2$

let's assume 2 $0<j<n$

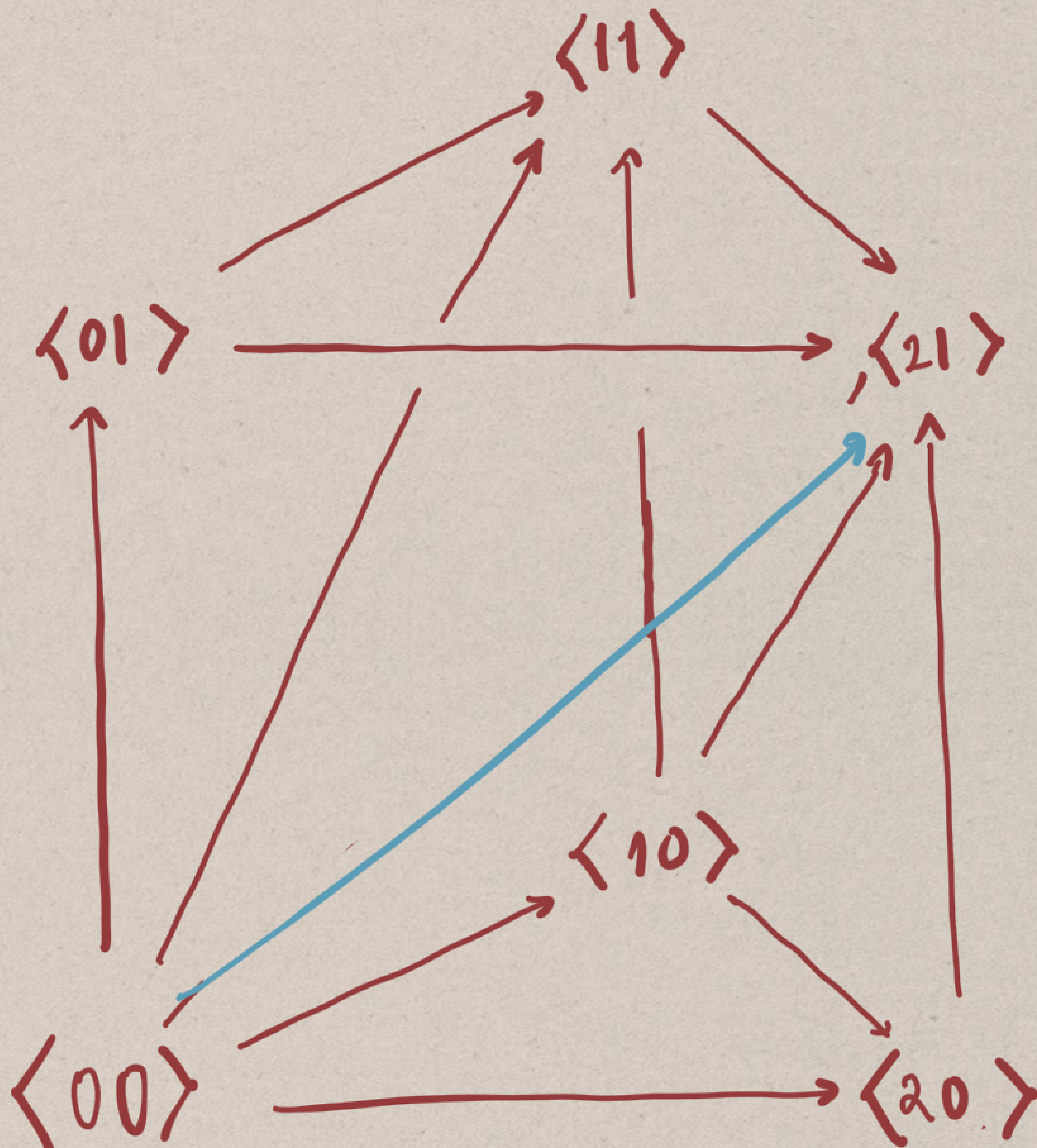
$$[k] \xrightarrow{(\beta,f)} [n]\times[2] \xrightarrow{r} [n]$$

$$i \longmapsto \beta(i) \text{ if } \beta(i)<j$$

$im(r\circ(\beta,f)) \subseteq \{0,\dots,j\}$

$\Delta^m \times \Delta^2$ (m=1)

↓ PRISM



m=1

$$\Delta^m \times \Lambda^2_1 \bigcup \partial\Delta^m \times \Delta^2 \subseteq \Delta^m \times \Delta^2$$

$\Lambda^2_1 \times \partial\Delta^m$

inner anodyne

then we show

$\overline{S}_3 \subseteq \overline{S}_1$

$S_1 = (\Lambda^n_k \subseteq \Delta^n)_{0<k<n}$

$S_2 = \overline{\mathrm{Cell}} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)$.

$S_3 = \mathrm{Cell} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)$

$S_4 = \overline{\mathrm{Cell}} \,\square\, S_1$

✓. $\overline{S_3} = \overline{\mathrm{Cell} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}$

$\subseteq \overline{\overline{\mathrm{Cell}} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}$

$\subseteq \overline{\overline{\mathrm{Cell}} \,\square\, \overline{(\Lambda^2_1 \subseteq \Delta^2)}}$

$\overline{\overline{\mathrm{Cell} \,\square\, (\Lambda^2_1 \subseteq \Delta^2}} = \overline{\mathrm{Cell} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}$

$\overline{S_1} \subseteq \overline{S_2} = \overline{S_3} = \overline{S_4}$

$\overline{S_3} \subseteq \overline{S_1}$

$\overline{S_1} \subseteq \overline{S_2}$

$\overline{S_1} = \cancel{\overline{\mathrm{Cell}} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}$

- $\overline{S_2} \subseteq \overline{S_4}$
- $\overline{S_1} \subseteq \overline{S_2} = \overline{\overline{\mathrm{Cell}} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}$ because of the retract lemma
- $\overline{S_4} = \overline{\overline{\mathrm{Cell}} \,\square\, S_1} \subseteq \overline{\overline{\mathrm{Cell}} \,\square\, \overline{\overline{\mathrm{Cell}} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}} \subseteq \overline{\overline{\mathrm{Cell}} \,\square\, \overline{\mathrm{Cell}} \,\square\, (\Lambda^2_1 \subseteq \Delta^2)}$

$$\begin{array}{ccc}
\overline{A \sqcap \overline{B}} & = & \overline{A \sqcap B} \\
\text{\rotatebox{90}{$\subseteq$}} & & \parallel \\
\overline{\overline{A} \sqcap \overline{B}} & \subseteq & \overline{\overline{A \sqcap B}}
\end{array}$$

$$\begin{array}{ccc}
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\text{\rotatebox{90}{$\subseteq$}} & & \parallel \\
\overline{\overline{A} \sqcap \overline{B}} & \subseteq & \overline{\overline{A \sqcap B}}
\end{array}$$

Joins & Slices

Join of Ordinary Categories :

$A, B \in \mathrm{Cat}$

$$A * B = \begin{cases} ob(A*B) : \\ \\ mor(A*B) : \end{cases}$$

Examples :

Exercises:

1. $\mathrm{Fun}(A * B, C) \simeq \left(f_A : A \to C,\ f_B : B \to C,\ \gamma : f_A \circ \pi_A \Rightarrow f_B \circ \pi_B \right)$

2. $\mathrm{Fun}(C, A * B) = \left(f : C \longrightarrow [1],\ f_{\{0\}} : C^{\{0\}} \longrightarrow A,\ f_{\{1\}} : C^{\{1\}} \longrightarrow B \right)$

Cones on categories

JOIN OF SIMPLICIAL SETS