Joins

A, B

$A * B \rightsquigarrow ob(A * B) = ob A \sqcup ob B$

$\rightsquigarrow Hom_{A*B}(x,y) = \begin{cases} Hom_A(x,y) & \text{if } x,y \in A \\ Hom_B(x,y) & \text{if } x,y \in B \\ * & \text{if } x \in A, y \in B \\ \emptyset & \text{if } x \in B, y \in A \end{cases}$

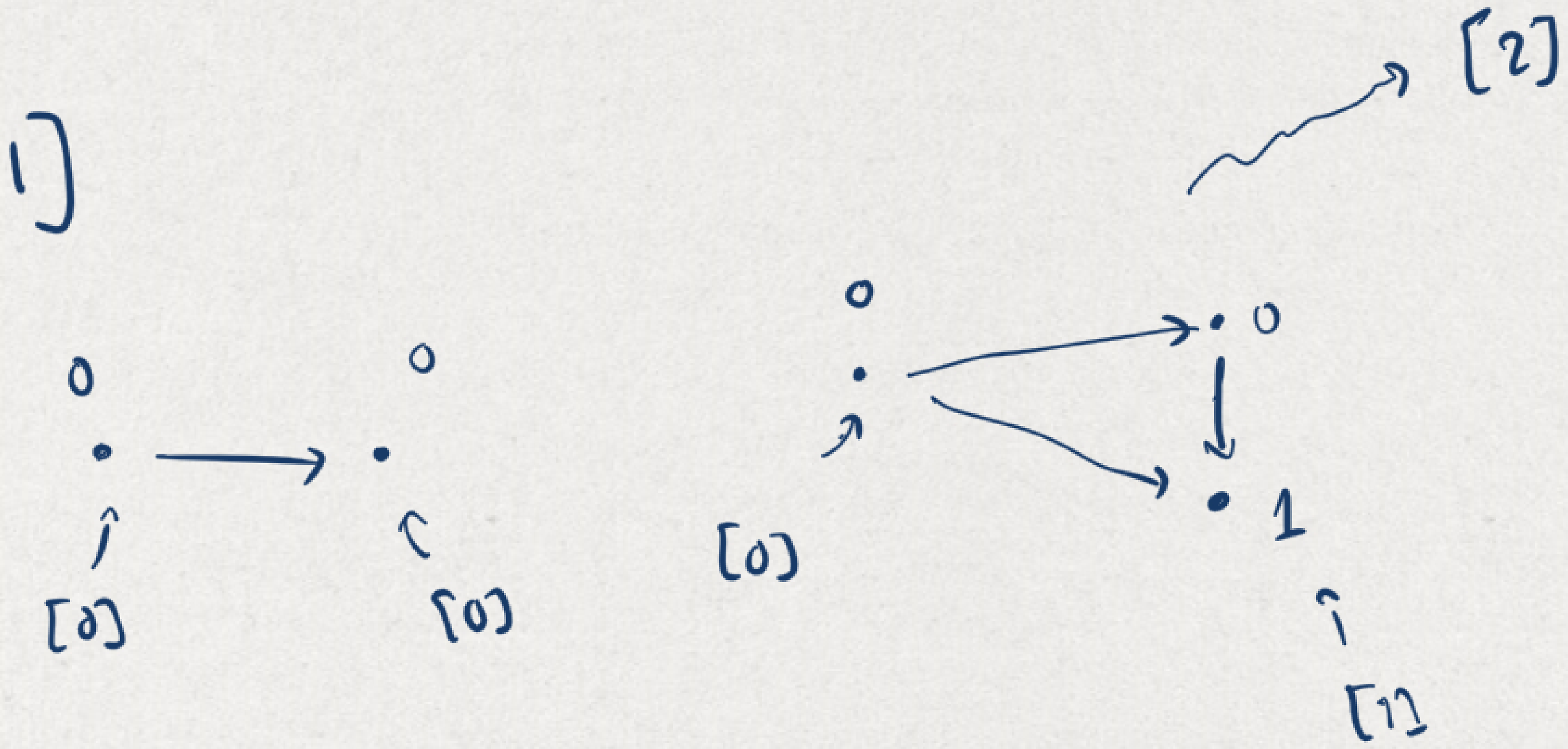
$A \hookrightarrow A * B$

$B \hookrightarrow A * B$

Ex: $[p] * [q] = [p+q+1]$

$[0] * [0] = [1]$

$[0] * [1] = [2]$



$[0] * [1]$

$[1] * [0]$

$f : A * B \longrightarrow \mathcal{C}$

cocones under the diagram

$P : A \to \mathcal{C}$

$(f_A : A \to \mathcal{C},\ f_B : B \to \mathcal{C},\ \gamma : f_A \pi_A \Rightarrow f_B \pi_B)$

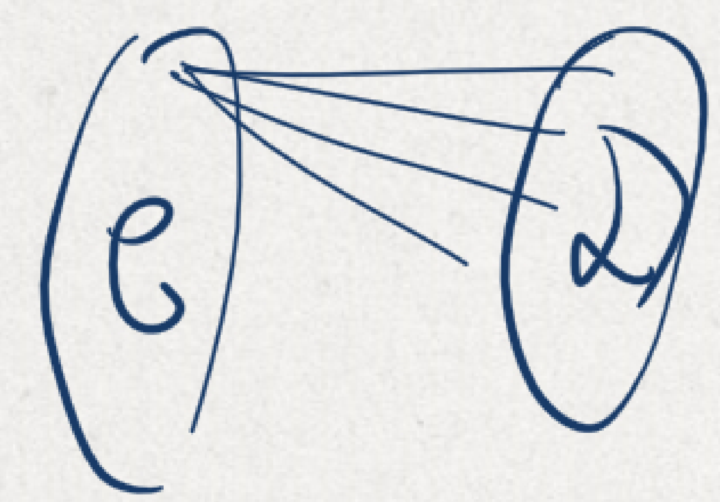
$$\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[dr] & A \arrow[d, hook] \\ & A * B \end{tikzcd} \quad , \quad \begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[dr] & B \arrow[d, hook] \\ & A * B \end{tikzcd}$$

$f : A \longrightarrow \mathcal{C} * \mathcal{D}$

$(\pi : A \longrightarrow [1],\ f^{\{0\}} : A^{\{0\}} \longrightarrow \mathcal{C},\ f^{\{1\}} : A^{\{1\}} \to \mathcal{D})$

$A^{\{0\}}$

$\mathcal{C} * \mathcal{D}$



$(A * B)^{op} \simeq B^{op} * A^{op}$

Cones on Categories

(Co)limits $\rightsquigarrow$ terminal objects in cone categories $\rightsquigarrow$ defining cones and cone categories $\dashrightarrow$ Slices

$[0] * \mathcal{C} \rightsquigarrow \mathcal{C}^{\triangleleft}$

left cone category (cones)

[0]

$\mathcal{C}$

$\mathcal{C} * [0] \rightsquigarrow \mathcal{C}^{\triangleright}$

right cone category (cocones)

$\mathcal{C}$

[0]

$p : \mathcal{A} \longrightarrow \mathcal{C}$ some ordinary diagram (functor)

$q : A^{\triangleright} \longrightarrow \mathcal{C} \Leftrightarrow q : A * [0] \longrightarrow \mathcal{C}$

$q|_A = p \qquad q\iota_A = p$

$\iota_A : A \hookrightarrow A * (0)$

1) $q(v) \in \mathcal{C}_0$

2) $a \in \mathcal{A} \quad q(a \to v) : p(a) = q(a) \longrightarrow q(v)$

3)
$$\begin{tikzcd} a \arrow[d, "\alpha"'] \\ a' \end{tikzcd} \Leftrightarrow \begin{tikzcd} p(a) \arrow[d, "q(\alpha)"'] \arrow[dr, "q(a\to v)"] & \\ p(a') \arrow[r, "q(a'\to v)"'] & q(v) \end{tikzcd}$$

$q : A^{\triangleright} \to \mathcal{C}$ s.t $q|_A = p$

$$\mathrm{Fun}_p(A^{\triangleright}, \mathcal{C}) \subseteq \mathrm{Fun}(A^{\triangleright}, \mathcal{C})$$

$$\mathrm{Colim}\ p \in \mathrm{Fun}_p(A^{\triangleright}, \mathcal{C})^{\text{initial}}$$

$$\left\{ \begin{tikzcd} S \arrow[r, "f"] \arrow[d, hook] & X \\ S * K \arrow[ur, dashed] & \end{tikzcd} \right\} \simeq \left\{ K \dashrightarrow X_{/f} \right\}$$

$$\mathrm{Hom}_{sSet_{/X}}(S * K, X) \simeq \mathrm{Hom}_{sSet}(K, X_{/f})$$

Category of cones

Joins For Simplicial Sets

Def: X, Y are simplicial sets

$$(X * Y)_n := \coprod_{[n] = [n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2}$$

$\cdot 0 \to \cdot 1 \to \cdot 2$

$$(X * Y)_0 = X_0 \sqcup Y_0$$

$$(X * Y)_1 = X_1 \sqcup X_0 \times Y_0 \sqcup Y_1$$

$$(X * Y)_2 = X_2 \sqcup X_1 \times Y_0 \sqcup X_0 \times Y_1 \sqcup Y_2$$

$\vdots$

$$X * (Y * Z) \simeq (X * Y) * Z$$

$$\begin{tikzcd} X \arrow[r] & X * Y \\ Y \arrow[ur] & \end{tikzcd}$$

Joins of Standard Simplicial Sets

$$\Delta^p * \Delta^q \simeq \Delta^{p+q+1}$$

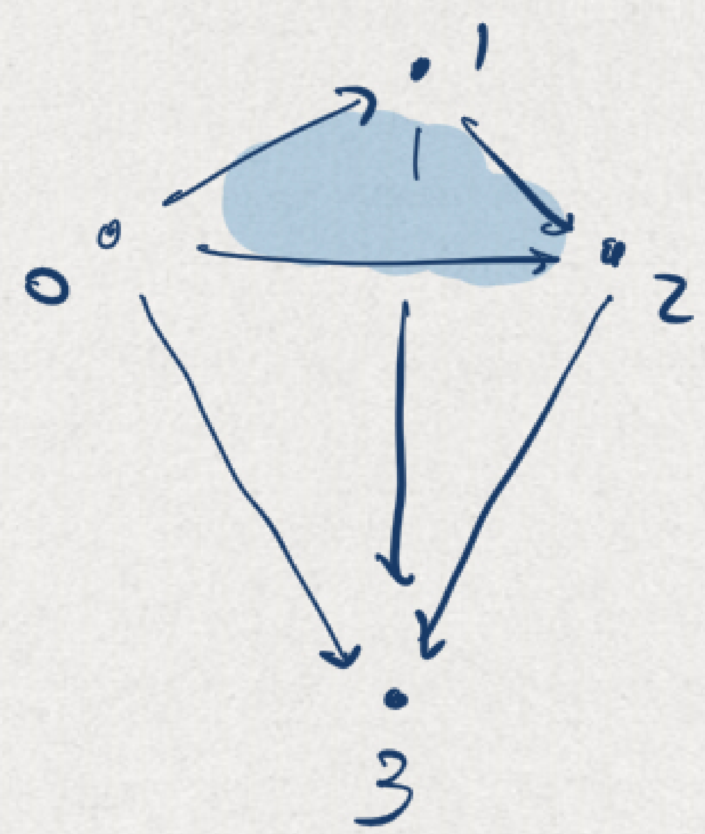
Left and Right Cones for Simplicial Sets

$$X * \Delta^0 := X^{\triangleright}$$

$$\Delta^0 * X := X^{\triangleleft}$$

$\rightsquigarrow$ $n = 2$

$$\to \quad (\partial \Delta^n)^{\triangleright} \simeq \Lambda^{n+1}_{n+1}$$



$\rightsquigarrow$ (

The join of ∞-categories is an ∞-category

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[ur, "g"] \arrow[rr, dashed] & & z \end{tikzcd}$$

Prop: Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, then $\mathcal{C} * \mathcal{D}$ is an ∞-category.

$$\Delta^n \simeq \Delta^1 * \Delta^{n-2}$$
$$\downarrow$$
$$\Delta^1$$

Proof:

1) **Lemma:** $f : K \to X * Y \Leftrightarrow (\pi : K \to \Delta^1, f^{\{0\}} : K^{\{0\}} \to X, f^{\{1\}} : K^{\{1\}} \to Y)$

$K^{\{j\}} = \pi^{-1}(j)$ $\quad \Lambda^n_j \to \mathcal{C} * \mathcal{D} \simeq (\pi' : \Lambda^n_j \to \Delta^1,$

$\pi'^{-1}(v) \to$ inner horn, standard simplex, empty

$$+ \quad \begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & \mathcal{C} * \mathcal{D} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

$\Delta^n \dashrightarrow \mathcal{C} * \mathcal{D} \simeq (\pi : \Delta^n \to \Delta^1, f^{\{0\}} : \pi^{-1}(0) \to \mathcal{C}, f^{\{1\}} : \pi^{-1}(1) \to \mathcal{D})$

$\pi^{-1}(v) \to$ empty, Standard simplex

Slices of ∞-Categories

Slices of 1-cats: $x \in \mathcal{C}_0$

$$\mathcal{C}_{/x} \rightsquigarrow \begin{tikzcd} c \arrow[r, "f"] \arrow[d, "g"'] & c \arrow[dl, "h"] \\ x & \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r] & c \\ x \arrow[u] \arrow[ur] & \end{tikzcd} \rightsquigarrow {}^{x/}\mathcal{C}$$

$x \downarrow \mathcal{C}$ $\quad$ $\mathcal{C} \downarrow x$

$$f : * \to \mathcal{C}$$
$$* \mapsto x$$

$$f : X \to \mathcal{C} \qquad \mathcal{C}_{/f} \dashrightarrow \{ c \xrightarrow{\gamma_x} f(x) \}_{x \in X}$$

$$\begin{tikzcd} & c' \arrow[d] \arrow[ddl, bend right, "\gamma'_x"'] \arrow[ddr, bend left, "\gamma'_{x'}"] & \\ & c \arrow[dl, "\gamma_x"] \arrow[dr, "\gamma_{x'}"'] & \\ f(x) \arrow[rr, "f(x \to x')"'] & & f(x') \end{tikzcd}$$

obj : $\tilde{f} : [0] * X \to \mathcal{C}$ s.t $\tilde{f}|_X = f$

mor : $\tilde{f} \to \tilde{f}'$ $\quad g : [1] * X \to \mathcal{C}$ s.t $g|_X = f$

$$g|_{\{0\} * X} = \tilde{f} \qquad g|_{\{1\} * X} = \tilde{f}'$$

Joins / Slice adjunction

$$X * - : sSet \longrightarrow sSet$$
$$Y \longmapsto X * Y$$
$$\emptyset \longmapsto X * \emptyset \simeq X$$

$$\begin{tikzcd} X \arrow[d] \\ X * Y \\ Y \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} X/sSet \\ X \arrow[u] \arrow[d, "id_X"'] \\ X \end{tikzcd}$$ is initial in $X/sSet$

$$X * - : sSet \longrightarrow X/sSet$$
$$Y \longmapsto (X \hookrightarrow X * Y)$$

$$\left\{ \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ X * K \arrow[ur, dashed] & \end{tikzcd} \right\} \simeq \left\{ K \dashrightarrow {}^{f/}\mathcal{C} \right\}$$

$K = \Delta^0$ $\qquad \Delta^0 \longrightarrow {}^{f/}\mathcal{C}$ $\qquad K = \Delta^1$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & \mathcal{C} \\ X * \Delta^0 \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

$\Delta^1 \longrightarrow {}^{f/}\mathcal{C} \rightsquigarrow$ 1-cells of ${}^{f/}\mathcal{C}$

$$\text{IS}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ X * \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

$\tilde{f} : X * \Delta^0 \longrightarrow \mathcal{C}$ s.t. $\tilde{f}\big|_X = f$