Iterated Segal Categories

Model for (∞, n)-categories

Segal categories: let K be sSet with Quillen model structure.

Δ: finite non-empty ordinals $[0], [1], \ldots$

Segal pre-category: $A : \Delta^{op} \to K$ such that $A[0]$ is discrete.

$$\begin{tikzcd} A[n](x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A[n] \arrow[d] \\ * \arrow[r, "x"'] & A[0]^{n+1} \end{tikzcd}$$

terminal object in K

$$A[n](x) = A(\underbrace{x_0, \ldots, x_n}_{\in A[0]})$$

Segal category: $A : \Delta^{op} \to K$ is Segal if $\forall n \in \mathbb{N}_{\geq 2}$, $\forall x \in A[0]^{n+1}$

$$\begin{tikzcd} A(x_0, \ldots, x_n) \arrow[r, "f"] \arrow[dr] & A(x_0, x_1) \times \ldots \times A(x_{n-1}, x_n) \arrow[d] \\ & A(x_i, x_{i+1}) \end{tikzcd}$$

f is a weak equivalence.

If M is a model category, we call it good if it is:

i) tractable; ii) left proper; iii) Cartesian.

Examples:

- Set is good;
- K is good;
- Func(D, Set)
  D connected

For X a set. View $\Delta \hookrightarrow \mathrm{Cat} \ni X$. We define

$$\Delta_X = \Delta/X = \begin{cases} \left(\mathrm{ob}: [n] \longrightarrow X\right) = \sum_{n=0}^{\infty} X^{n+1} \\ \text{morphisms: } \begin{tikzcd} {[n]} \arrow[rr, "f"] \arrow[dr, "x"'] & & {[m]} \arrow[dl, "y"] \\ & X & \end{tikzcd} \end{cases}$$

In particular

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & {[1]} \arrow[l, "\sigma_0" description] \end{tikzcd}$$

$$(x) \xrightarrow{\delta_1} (x,y) \qquad (y) \xrightarrow{\delta_0} (y,x) \qquad (x,x) \xrightarrow{\sigma_0} (x)$$

Func(Δ_X^{op}, M),

$$\begin{tikzcd} & A(x) \\ A(x,y) \arrow[ur, "A\delta_1"] \arrow[dr, "A\delta_0"'] & \\ & A(y) \end{tikzcd}$$

$$A(x) \xrightarrow{A\sigma} A(x,x)$$

$PC(M,X) \longrightarrow$ pre-categories with vertex set X enriched in a good model cat. M

$A \in Func(\Delta_X, M)$ is a pre-cat if $A(x)$ is terminal for all $x \in X$. We have

$$PC(M,X) \underset{\text{full}}{\hookrightarrow} Func(\Delta_X^{op}, M).$$

$$PC(M) = \int_{X \in Set} PC(M,X)$$

where $PC(M,-) : \underline{Set}^{op} \longrightarrow CAT$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \Delta_X \arrow[d, "\Delta_f"'] \\ \Delta_Y \end{tikzcd} , \qquad \begin{tikzcd} {[n]} \arrow[r] & X \arrow[d, mapsto] \\ {[n]} \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\begin{tikzcd} Func(\Delta_X^{op}, M) & PC(M,X) \arrow[l, hook'] \\ Func(\Delta_Y^{op}, M) \arrow[u, "(\Delta_f^{op})^*"] & PC(M,Y) \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

$PC(M)$:

- objects: $(X, A \in PC(M,X))$
- morphisms:

$$(X,A) \longrightarrow (Y,B) = (X \xrightarrow{f} Y,\ A \longrightarrow \Delta_f^{op}(B))$$

Let us write $PC^0(M) = M$, $PC^{n+1}(M) = PC(PC^n(M))$.

$A \in PC^{n+1}(M)$ is a full Segal object if all Segal maps $A(x_0, \dots, x_n) \to A(x_0, x_1) \times \dots \times A(x_{n-1}, x_n)$ are weak equivalences and all $A[n] \in PC^n(M)$ are full Segal objects.