Higher Category Theory

$$\mathrm{Alg}_n(\mathcal{S}) \longrightarrow \text{sym. monoidal } (\infty,1)\text{-category}$$

TQFT: $(Z: \mathrm{Cob}_n^{\otimes} \longrightarrow \mathrm{Vect}^{\otimes})$

Fully extended TQFT: $Z: \mathrm{Bord}_n^{fr} \longrightarrow \mathcal{C}^{\otimes n}$

Cobordism hypothesis: Z is fully determined by $Z(*)$.

We aim to construct a TQFT $Z: \mathrm{Cob}_n^{fr} \to \mathrm{Alg}_n(\mathcal{S})$.

Terminology:

- Mfld:
  - objects: n-dimensional framed manifolds;
  - morphisms: embeddings;
  - symmetric monoidal: disjoint union $\amalg$.
- $\mathrm{Disk}_n^{fr,\amalg} \hookrightarrow \mathrm{Mfld}$
- Symmetric monoidal functors

$$\mathrm{Disk}_n^{fr,\amalg} \xrightarrow{A} \mathcal{S}^{\otimes}$$

are E_n-algebras.

Example: For $n=2$ and $S=\mathrm{Cat}$, then E_2-algebras are braided monoidal categories.
For $n=1$ and Ch_K, then E_1-algebras are A_∞-algebras.
For general n, with $S=\mathrm{Top}_*$, E_n-algebras are group-like n-fold loop spaces.

$$\begin{tikzcd} \mathrm{Disk}_n^{fr,\amalg} \arrow[r, "A"] \arrow[d] & S^{\otimes} \\ \mathrm{Mfld}^{fr,\amalg} \arrow[ur, dashed, "\substack{\text{left Kan}\\ \text{extension}\\ \text{(sym. monoidal)}}"'] & \end{tikzcd} \quad \rightsquigarrow \quad \int A : \mathrm{Mfld}^{fr,\amalg} \to S$$

factorization homology with coefficients in A

$$\int_M A = \operatorname*{colim}_{\mathrm{Disk}^{fr,\amalg}/M} A$$

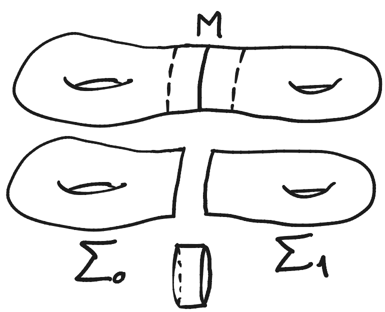
Ayala - Francis
Lurie
Morison - Walker
Beilinson - Drinfeld

Theorem ($\otimes$-excision): For a framed n-manifold Σ, decompose

$$\Sigma \simeq \Sigma_0 \bigcup_{M\times\mathbb{R}} \Sigma_1$$

$$\Sigma_0 \hookrightarrow \Sigma\,, \quad \Sigma_1 \hookrightarrow \Sigma$$

$$\int_\Sigma A \simeq \int_{\Sigma_0} A \otimes_{\int_{M\times\mathbb{R}}} \int_{\Sigma_1} A\,.$$



One can:

- add manifold structures;
- stratified manifolds
  - $\nearrow$ coefficients: E_n-algs + bimodules
  - $\searrow$ coefficients: (∞,n)-categories.

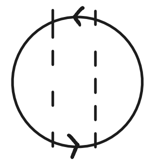
$$M\times\mathbb{R} \amalg M\times\mathbb{R} \hookrightarrow M\times\mathbb{R}$$

$$\mathbb{R}\amalg\mathbb{R} \hookrightarrow \mathbb{R}$$

$$\begin{tikzcd}
\int_{M\times\mathbb{R}} A \arrow[dr] & \otimes & \int_{M\times\mathbb{R}} A \arrow[dl] \arrow[r] & \int_{M\times\mathbb{R}} A \\
& E_1\text{-algebras} & &
\end{tikzcd}$$

Example:

$$\int_{\circlearrowleft} A \simeq \int_{\leftarrow} A \otimes_{\int_{\rightleftarrows} A} \int_{\rightarrow} A \simeq A \otimes_{A\otimes A^{op}} A^{op}$$



We have a TQFT

$$n\mathrm{Cob} \xrightarrow{Z=\int A} \mathrm{Alg}_1(\mathcal{S})$$

$$M \longmapsto \int_{M\times\mathbb{R}} A \dashrightarrow J \longleftarrow \text{pick this}$$

$$\Sigma \longmapsto \int_{\Sigma} \mathbb{R}$$

Dip into factorization algebras:

Definition: Let M be a manifold. A discrete colored operad $\mathrm{Disk}(M)$ with colors open subsets $\simeq (\mathbb{R}^n)^{\amalg k}$ and sets of maps

$$\mathrm{Discs}(U_1,\dots,U_n;V) = \begin{cases} * & \text{if } U_1\amalg\dots\amalg U_n \hookrightarrow V \\ \emptyset & \text{otherwise} \end{cases}$$

Definition: A prefactorization algebra in $\mathcal{S}$ is a $\mathrm{Disc}(M)$-algebra valued in $\mathcal{S}$.

Definition: A factorization algebra $\mathcal{F}$ is a prefactorization algebra if

i) $\mathcal{F}(U_1) \otimes \ldots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(U_1 \amalg \ldots \amalg U_n)$ is an equivalence;

ii) Codescent for Weiss covers.

Example: $\mathrm{Disk}_n^{fr} \simeq \mathrm{Disk}_n^{fr}[(D_1 \hookrightarrow D_2)^{-1}]$

Given an E_n-algebra, one can produce a factorization algebra using factorization homology

$$\mathcal{U} \longrightarrow \int_{\mathcal{U}} A \in S.$$

Theorem [Lurie]: $N(\mathrm{Disk}(\mathbb{R}^n))[(D_1 \hookrightarrow D_2)^{-1}] \simeq E_n$.

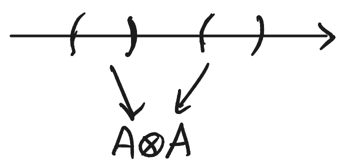
$$E_n\text{-algebras} \longleftrightarrow \begin{array}{c}\text{locally constant} \\ \text{fact. algebra in } \mathbb{R}^n\end{array}.$$

Definition: A locally constant factorization algebra valued in S is a prefactorization algebra $\mathcal{F}: \mathrm{Discs}(M) \longrightarrow S$ s.t

$$\mathcal{F}(D_1) \simeq \mathcal{F}(D_2) \text{ if } D_1 \hookrightarrow D_2.$$

Example:

$$\text{Algebras} \Leftrightarrow \begin{array}{c}\text{locally constant} \\ \text{factorization algebras} \\ \text{on } \mathbb{R}\end{array}$$

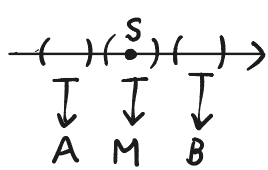


Example:

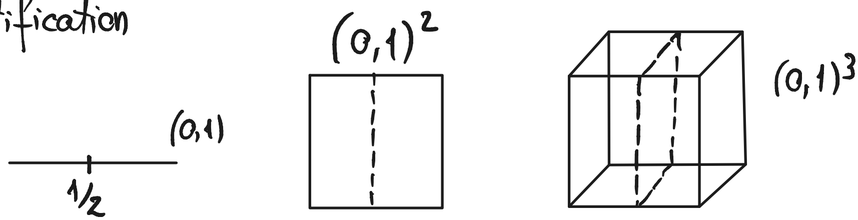
M → (A,B)-bimodule

A B

not locally constant



Stratification



Definition: A constructible factorization algebra $\mathcal{F}$ is a fact. algebra on $(0,1)^n$ s.t is locally constant with respect to $D_1 \hookrightarrow D_2$ same standard disks.

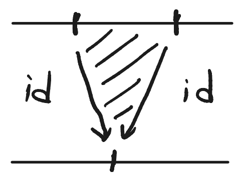
Definition: $p: M \to N$, $\mathcal{F}$ a fact. alg. on M can be pushed-forward to a factorization alg. on N by

$$p_* \mathcal{F}(U) := \mathcal{F}(p^{-1}(U))$$

If p is a fiber bundle, p_* preserves local constancy.

Definition:

If p → local diffeomorphisms / refinement of stratifications / collapse-rescale.



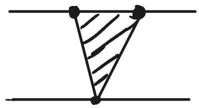
Segal object in Cat_∞ $\Delta^{op} \to \mathrm{Cat}_\infty$

restriction on left or right of point

$$X_0 = \left(\begin{array}{c} (\infty,1)\text{-cat of locally} \\ \text{constant fact. alg on } \mathbb{R} \end{array} \right)$$

$$s \uparrow \uparrow t$$

$$X_1 = \left(\begin{array}{c} (\infty,1)\text{-cat of constructible} \\ \text{fact. alg on } (\mathbb{R}, \text{—•—}) \end{array} \right)$$



$\uparrow\uparrow\uparrow$

$$X_2 = \left(\begin{array}{l} (\infty,1)\text{-cat of constructible} \\ \text{fact. alg. on } (\mathbb{R}, \quad) \end{array} \right)$$