Weighted ~~(co)limits~~ in $(\infty,1)$-cats

[Based on Rovelli '14]

§1: Ordinary category case

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Recall: $Ob(\mathcal{I} * \mathcal{J}) = Ob(\mathcal{I}) \amalg Ob(\mathcal{J})$, $Ar(\mathcal{I} * \mathcal{J}) \cong Ar(\mathcal{I}) \amalg Ar(\mathcal{J}) \amalg (Ob(\mathcal{I}) \times Ob(\mathcal{J}))$

Comp$^{\text{on}}$! determined by saying $\mathcal{I}, \mathcal{J}$ subcats of $\mathcal{I} * \mathcal{J}$

$- * \mathcal{J} : \underline{Cat} \to {}^{\mathcal{J}}/\underline{Cat}$ has radj $-/_- : {}^{\mathcal{J}}/\underline{Cat} \to \underline{Cat}$,

$d : \mathcal{J} \to \mathcal{C} \longmapsto \mathcal{C}/d$. $Ob(\mathcal{C}/d) \cong Hom_{\underline{Cat}}([0], \mathcal{C}/d) \cong Hom_{{}^{\mathcal{J}}/\underline{Cat}}([0] * \mathcal{J}, \mathcal{C})$,

$Ar(\mathcal{C}/d) \cong Hom_{{}^{\mathcal{J}}/\underline{Cat}}([1] * \mathcal{J}, \mathcal{C})$.

Why is $i \in \mathcal{I}, j \in \mathcal{J}$, is $Hom_{\mathcal{I} * \mathcal{J}}(i, j) = \{*\}$? Can it be something else?

Answer: yes, if we're careful.

Def$^{\text{n}}$ A f$^{\text{r}}$ $p : \mathcal{E} \to \mathcal{B}$ is a discrete fibration on $\mathcal{B}$, if for each $e \in \mathcal{E}$ & arrow $f : b' \to pe \in \mathcal{B}$, $\exists!$ lift $g : e' \to e$ (i.e. $p(g) = f$).

Idea: Let $e \in p^{-1}(b)$. Then $id_b : b \to b = pe$ has a at most one ~~unique~~ lift with codomain e, which must be id_e. Let E_b the $Ob(E_b) = Ob(p^{-1}(b))$, $p(Ar(E_b)) = id_b$.

Then the E_b are discrete categories, i.e. sets. Indeed,

Thm (Loregian, Riehl) $\underline{DFib}(\mathcal{B}) \cong [\mathcal{B}^{op}, \underline{Set}]$.

In particular, $\forall p : \mathcal{B} \to \mathcal{B}$, have $p^{-1}(-) : \mathcal{B} \to \underline{Set}$, a functor denote W_p

E.g. $P = Id : \mathcal{B} \to \mathcal{B}$, $W_p(b) = \{b\}$, i.e. $W_p \cong * : \mathcal{B} \to \underline{Set}$

Note to self: Have you done pushout yet?

[N.B. Haven't seen that pushouts exist in $\underline{sSet}$, but will be assuming it]

Def^n Let $p : \tilde{\mathcal{J}} \to \mathcal{J}$ be a discrete fibration. The <u>join</u> of a cat $\mathcal{I}$ ~~weighted~~ <u>weighted</u> by $p : \tilde{\mathcal{J}} \to \mathcal{J}$ is $\mathcal{I} *^p \mathcal{J}$, where:

- $Ob(\mathcal{I} *^p \mathcal{J}) \cong Ob(\mathcal{I}) \amalg Ob(\mathcal{J})$.
- $i, i' \in \mathcal{I}$, $Hom_{\mathcal{I} *^p \mathcal{J}}(i, i') = Hom_{\mathcal{I}}(i, i')$
- $j, j' \in \mathcal{J}$, $Hom_{\mathcal{I} *^p \mathcal{J}}(j, j') = Hom_{\mathcal{J}}(j, j')$
- $i \in \mathcal{I}, j \in \mathcal{J}$ $Hom_{\mathcal{I} *^p \mathcal{J}}(j, i) = \emptyset$
- $i \in \mathcal{I}, j \in \mathcal{J}$, $Hom_{\mathcal{I} *^p \mathcal{J}}(i, j) = W_p(j)$,

i.e., $|W_p(j)|$ many arrows from each i to j.

<u>Rmk</u>: Can be written as pushout of cats:

$$\mathcal{I} *^p \tilde{\mathcal{J}} \cong (\mathcal{I} * \tilde{\mathcal{J}}) \amalg^p_{\tilde{\mathcal{J}}} \mathcal{J}$$

(for
$$\begin{tikzcd} & \tilde{\mathcal{J}} \arrow[dl, hook'] \arrow[dr, "p"] & \\ \mathcal{I} * \tilde{\mathcal{J}} \arrow[dr, "r"'] & & \mathcal{J} \arrow[dl, "s"] \\ & K & \end{tikzcd}$$
a cone, def $\sigma : \mathcal{I} *^p \mathcal{J} \to K$ by $\sigma(i) = r(i)$, $\sigma(j) = s(j)$, $\sigma(*_{\tilde{j}} : i \to p(\tilde{j})) = r(* : i \to \tilde{j})$)

Composition: $\mathcal{I}, \mathcal{J}$ subcategories. For $f : i \to i'$, $s : i' \to j$, $sf = s$

For $s : i \to j$, $g : j \to j'$, recall $W_p : \mathcal{J} \to \underline{Set}$ a functor, so $W_p(g) : W_p(j) \to W_p(j')$.
But $W_p(j) = Hom_{\mathcal{I} *^p \mathcal{J}}(i, j)$ and $W_p(j') = Hom_{\mathcal{I} *^p \mathcal{J}}(i, j')$, so we get
$gs = W_p(g)(s)$.

We can again define $- *^p \mathcal{J} : \underline{Cat} \to {}^{\mathcal{J}/}\underline{Cat}$.

<u>Psn</u> $- *^p \mathcal{J}$ admits a right adjoint $-^p/_- : {}^{\mathcal{J}/}\underline{Cat} \to \underline{Cat}$, $d : \mathcal{J} \to \mathcal{C} \mapsto \mathcal{C}^p/_d$,
<u>weighted slice</u>.

Def^n Explicitly, $Ob(\mathcal{C}^p/_d) \cong Hom_{\mathcal{J}/\underline{Cat}}([0] *^p \mathcal{J}, \mathcal{C})$,

$$Ar(\mathcal{C}^p/_d) \cong Hom_{\mathcal{J}/\underline{Cat}}([1] *^p \mathcal{J}, \mathcal{C}).$$

Note: notate W_- $DFib(\mathcal{J}) \xrightarrow{\sim} [\mathcal{J}^{op}, \underline{Set}] : el_{\mathcal{J}}$.
Let $d : \mathcal{J} \to \mathcal{C}$ a diagram, $W : \mathcal{J} \to \underline{Set}$ a weight. Then $\mathcal{C}^{el_W}/_d$ is the category of weighted cones in sense of weighted limits.

Weighted cones in quasi-cats

Reminder: in $sSet$, have join $I*J$, where $(I*J)_n \cong \cancel{\coprod_{i=-1}^{n}(I_i\times J_{n-1-i})}\ \coprod_{i=0}^{n-1}(I_i\times J)_{n-1-i})\sqcup I_n\sqcup J_n$

& $-*J : sSet \to J/sSet$ admits a r. adj $-/_- : J/sSet \to sSet$, where $(Q/_d)_n \cong Hom_{sSet}(\Delta^n, Q/_d) \cong Hom_{J/sSet}(\Delta^n * J, Q)$

Defⁿ Let $p:\tilde J \to J$ be a simplicial map. The join of I & J weighted by $p:\tilde J\to J$ is the simp set $I*^pJ = (I*\tilde J)\sqcup^p_{\tilde J}J$. Levelwise, this is $(I*^pJ)_n \cong I_n\sqcup J_n\sqcup\left(\coprod_{i=0}^{n}(I_i\times\tilde J_{n-1-i})\right)$.

~~Indeed, as~~ As before, this defines a functor

$$-*^pJ : sSet \xrightarrow{-*\tilde J} \tilde J/sSet \xrightarrow{-\sqcup^p_{\tilde J}J} J/sSet.$$

Prop The weighted join $-*^pJ$ admits a right adjoint

$$-^p/_- : J/sSet \xrightarrow{-\circ p} \tilde J/sSet \xrightarrow{-/_-} sSet$$

Pf Know already that $-*\tilde J$ and $-/_-$ are adjoint.

Note ~~...~~ $-\sqcup^p_{\tilde J}J$ is well-defined: $\tilde J\sqcup^p_{\tilde J}J = J$ via

$$\begin{tikzcd} & \tilde J \arrow[dl, "id"'] \arrow[dr, "p"] & \\ \tilde J \arrow[dr, "p"'] & & J \arrow[dl, "id"] \\ & J & \end{tikzcd}$$

. WTS $Hom_{J/sSet}(J\xrightarrow{d\sqcup^p_{\tilde J}J}X\sqcup^p_{\tilde J}J,\ J\xrightarrow{m}Y)\cong Hom_{\tilde J/sSet}(\tilde J\xrightarrow{d}X,\ \tilde J\xrightarrow{mp}Y)$.

If $r\in Hom_{\tilde J/sSet}(\tilde J\xrightarrow{d}X,\ \tilde J\xrightarrow{mp}Y)$, then $r:X\to Y$ ST $mp = rd$, i.e.

$$\begin{tikzcd} & \tilde J \arrow[dl, "d"'] \arrow[dr, "p"] & \\ X \arrow[dr] \arrow[ddr, "r"'] & & J \arrow[dl, "d\sqcup J"'] \arrow[ddl, "m"] \\ & X\sqcup^p_{\tilde J}J \arrow[d, dashed] & \\ & Y & \end{tikzcd}$$

But by defⁿ of $X\sqcup^p_{\tilde J}J$, $\exists!\ \bar r : X\sqcup^p_{\tilde J}J\to Y$ ST, in particular, $\bar r(d\sqcup^p_{\tilde J}J) = m$, ~~i.e.~~ as required.

~~Conversely~~, given

Conversely, given s : $X \amalg^p_{\tilde{J}} J \to Y$ st $s(d \amalg^p_{\tilde{J}} J) = m$, have comm^{ve} diagram

$$\begin{tikzcd} & \tilde{J} \arrow[dl, "d"'] \arrow[dr, "p"] & \\ X \arrow[dr, "X \amalg^p_{\tilde{J}} p"'] & & J \arrow[dl, "d \amalg^p_{\tilde{J}} J"'] \arrow[d, bend left, "m"] \\ & X \amalg^p_{\tilde{J}} J \arrow[r, "s"] & Y \end{tikzcd}$$

But by def^n of pushout, s must be the ! such m^{sm}, giving bijection. □

<u>Def^n</u> The <u>quasi-category of weighted cones</u> in Q over the diagram $d : J \to Q$ is $Q^p_{/d}$, and is ! up to iso^{sm} by adj^n.
Explicitly, $(Q^p_{/d})_n \cong \mathrm{Hom}_{sSet}(\Delta^n, Q^p_{/d}) \cong \mathrm{Hom}_{J/sSet}(\Delta^n *^p J, Q)$.

E.g. Let $\tilde{J}$ be s. set

$$\begin{tikzcd} \tilde{0}' \arrow[r] & \tilde{2} & \tilde{0} \arrow[l] \arrow[dl] \\ & \tilde{1} \arrow[u] & \end{tikzcd} = \Delta^1 \amalg_{\Delta^0} \Delta^2,$$

Let J be the s. set $0' \longrightarrow 1 \longleftarrow 0 = \Delta^1 \amalg_{\Delta^0} \Delta^1 \cong \Lambda^2_2$

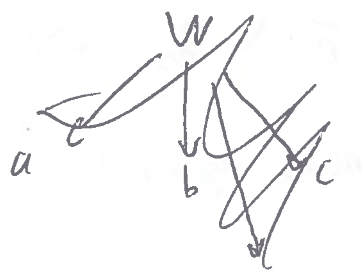
and let $p : \tilde{J} \to J$ be given by sending $\tilde{1}$ and $\tilde{2}$ to 1.

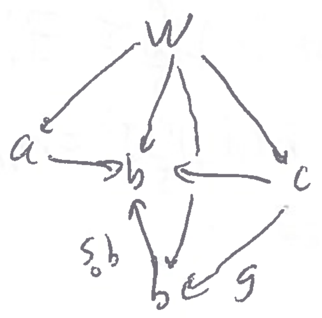
Let $d : \Lambda^2_2 \to Q$ in a quasi-cat Q with image $a \xrightarrow{f} b \xleftarrow{g} c$, 1-simplex
head diagram

A cone over d is a vertex k with two 2-simplices glued on ~~an edge~~:

$$\begin{tikzcd} & k \arrow[dl] \arrow[d] \arrow[dr] & \\ a \arrow[r, "f"'] & b & c \arrow[l, "g"] \end{tikzcd}$$

A weighed cone over d by p is a 2-simplex glued to a 3-simplex along a 1-simplex:





Weighted limits:

A cone for a functor $F: J \to \mathcal{C}$ ~~over~~ under an object $c \in \mathcal{C}$ assigns to each $j \in J$ an arrow $c \to F(j)$ with coherence conditions. Let $*: J \to \underline{Set}$ be the constant functor ~~to~~ to the one element set $\{*\}$. Then

Psn A cone of F under ~~over~~ c is a n.t. $* \to \mathcal{C}(c, F(-))$.

Pf for each $j \in J$, $* \to \mathcal{C}(c, F(j))$ picks out single m^sm $c \to F(j)$. ~~Need~~ Need

$$\begin{tikzcd} * \arrow[r, "id"] \arrow[d] & * \arrow[d] \\ \mathcal{C}(c, F(j)) \arrow[r, "F(f)\circ -"'] & \mathcal{C}(c, F(j')) \end{tikzcd}$$

to commute, ~~...~~ giving coherence. □

Psn For $\lim F$ the limit of F,

$$\mathcal{C}(c, \lim F) \cong [J, \underline{Set}](*, \mathcal{C}(c, F)) \quad \forall c \in \mathcal{C}.$$

Pf (sketch) If we have a cone n under c, by def^n of $\lim F$ ∃! m^sm $c \to \lim F$ comp. with cones. Conversely, given a m^sm $f: c \to \lim F$, can compose with cone ~~of lim~~ under $\lim F$ to get cone under c □.

~~Weighted~~ Weighted limit: Just replace $*: J \to \underline{Set}$ with some other $W: J \to \underline{Set}$*. Then limit of F weighted by W is $\lim^W F$ ST

$$\mathcal{C}(c, \lim^W F) \cong [J, V](W, \mathcal{C}(c, F)) \quad \forall c \in \mathcal{C}.$$

* or ~~any~~ $W: J \to V$ for any V-enriched cat $\mathcal{C}$

Weighted limits in Q-cats

Keep

Let Q an $(\infty\text{-}1)$-cat. A cone ~~under~~ over the diagram $d: J \to Q$ is an ~~Q-~~ ~~vertex~~ ~~object $a \in Q$~~ vertex $a \in Q$ w a limit cone $l: \Delta^0 \to Q_{/d}$ from a to d. This is the same as a map $l: \Delta^0 * J \to Q$ ST

$$\begin{tikzcd} J \arrow[d, hook] \arrow[dr, "d"] & \\ \Delta^0 * J \arrow[r, "l"] & Q \\ \Delta^0 \arrow[u, hook] \arrow[ur, "a"'] & \end{tikzcd}$$

~~Weighted limits~~ in ~~Q-cats~~ commutes

Def^n Terminal object of an ∞-cat Q is a vertex t with the following lifting property $\forall n$:

$$\begin{tikzcd} \Delta^0 \arrow[r, "n"] \arrow[rr, bend left, "t"] & \partial\Delta^n \arrow[r] \arrow[d, hook] & Q \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Def^n The limit cone of a diagram $d: J \to Q$ is a vertex $\lim d$ & a lim cone $l: \Delta^0 * J \to Q$ from $\lim d$ to d which is terminal in $Q_{/d}$.

(lifting prop
$$\begin{tikzcd} \Delta^0 \arrow[r, "n"] \arrow[rr, bend left, "l"] & \partial\Delta^n \arrow[r] \arrow[d, hook] & Q_{/d} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$
)

Now, let $p: \hat{J} \to J$ be a weighting. We do the obvious thing.

Def^n The limit of $d: J \to Q$ weighted by p is a v^x $\lim^p d$ of Q w/ a weighted cone $l: \Delta^0 *^p J \to Q$ from $\lim^p d$ to d that is terminal in $Q^p_{/d}$.

(weighted cone: ~~map~~ ~~...~~ over v^x a: map $l: \Delta^0 *^p J \to Q$ ST

$$\begin{tikzcd} J \arrow[d, hook] \arrow[dr, "d"] & \\ \Delta^0 *^p J \arrow[r, "l"] & Q \\ \Delta^0 \arrow[u, hook] \arrow[ur, "a"'] & \end{tikzcd}$$

commutes)

Thm Let $p: \hat{J} \to J$ a weight. Then $\lim^p d \cong \lim(d \circ p: \hat{J} \xrightarrow{p} J \xrightarrow{d} Q)$.