Chiral CFT

Reference: André Henriques

$$Z : \mathrm{Cob}^{\mathrm{conf}} \longrightarrow \begin{array}{l}\text{Concrete Linear} \\ \text{Categories}\end{array}$$

Topological setting: diffeomorphism classes of $\Sigma : M \to N$ with identity $[M \times I]$.

Conformal setting: adapt the bordism category.

Conformal structure: (Σ, g) where $g \sim g'$ if there exists $\lambda : \Sigma \longrightarrow \mathbb{R}$ such that $g = \lambda^2 g'$.

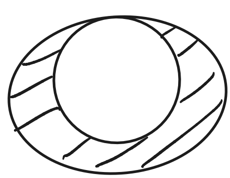
In 2 dimensions:

orientable mflds with conformal structure = Riemann surfaces.

We have the category Cob_2 with:

- objects: disjoint unions of S^1;
- morphisms: complex cobordisms;

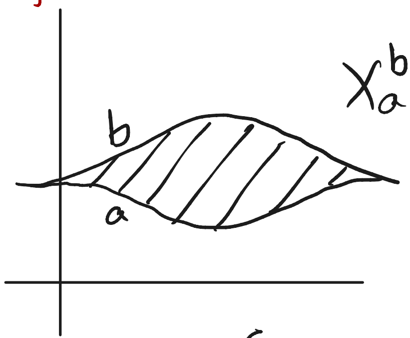
To add identities - cobordism with thin parts:



Definition:

$$a, b : \mathbb{R} \to \mathbb{R}, \quad a \leq b$$

$$X_a^b = \{ x + iy \in \mathbb{C} \mid a(x) \leq y \leq b(x) \}$$



$$\mathcal{O}_{X_a^b}(U) = \left\{ f : U \to \mathbb{C} \;\middle|\; \begin{array}{l} f|_{U \cap \underbrace{\mathring{X}_a^b}_{\text{interior}}} \text{ holomorphic} \\ \exists V \subseteq \mathbb{C} \text{ and } g \in C^\infty(V) \\ \text{such that } f = g|_U \end{array} \right\}$$

$(X_a^b, \mathcal{O}_{X_a^b})$ is the local model.

Definition: A complex bordism with thin parts is a ringed space $(\Sigma, \mathcal{O}_\Sigma)$ equipped with subspaces $\partial_{in} \Sigma \subseteq \Sigma$ and $\partial_{out} \Sigma \subseteq \Sigma$ (not necessarily disjoint) which is locally isomorphic to $(X_a^b, \mathcal{O}_{X_a^b})$ for some $a \leq b$, and $\partial_{in}\Sigma, \partial_{out}\Sigma$ are locally $\partial_{in} X_a^b, \partial_{out} X_a^b$.

Given oriented 1-manifolds S_1, S_2, a bordism with thin parts $\Sigma : S_1 \Rightarrow S_2$ is a diffeomorphism $(\Sigma, \mathcal{O}_\Sigma, \varphi_{in}, \varphi_{out})$ with $\varphi_{in} : S_1 \to \partial_{in}\Sigma$, $\varphi_{out} : S_2 \to \partial_{out}\Sigma$.

For $\Sigma : S_1 \Rightarrow S_2$ and $\Sigma' : S_2 \Rightarrow S_3$ bordisms with thin parts, then conformal welding gives

$$(\Sigma \cup_{S_2} \Sigma', \mathcal{U}_{\Sigma \underset{S_2}{\cup} \Sigma'}) : S_1 \Rightarrow S_3$$

which is a again a bordism with thin parts.

A chiral CFT is a symmetric monoidal functor

$$\mathrm{Cob}^{conf} \longrightarrow \underset{\substack{\cup \\ \mathrm{Rep}(A),\ \mathcal{U}: \mathrm{Rep}(A) \longrightarrow \mathrm{TopVect}}}{\text{Conc. Lin. Cat.}} + \text{holomorphicity condition.}$$

$$S \longmapsto \underset{\substack{\| \\ \mathrm{Rep}(A(S))}}{\varphi(S)}$$

$\mathrm{Rep}(A(S)) \to A(S)$ algebra of observables

$$\begin{tikzcd}
\Sigma: S_1 \Rightarrow S_2 \longmapsto \varphi(S_1) \arrow[rr, "F_\Sigma"] \arrow[dr, "\mathcal{U}_1"'] & & \varphi(S_2) \arrow[dl, "\mathcal{U}_2"] \\
& \mathrm{TopVect} &
\end{tikzcd}$$

$$F_\Sigma = H_\Sigma \underset{A(\partial_{in}\Sigma)}{\otimes} - : \mathrm{Rep}(A(S_1)) \longrightarrow \mathrm{Rep}(A(S_2)).$$

$H_\Sigma \to$ bimodule

For every closed compact oriented 1-manifolds $\varphi(S)$ linear category	$\mathcal{U}: \varphi(S) \longrightarrow \mathrm{TopVect}$ faithful
For every complex bordism $\Sigma: S_1 \Rightarrow S_2$ $F_\Sigma: \varphi(S_1) \longrightarrow \varphi(S_2)$	For $\lambda \in \varphi(S_1)$ a linear map $Z_\Sigma: \mathcal{U}(\lambda) \to \mathcal{U}(F_\Sigma(\lambda))$

For every $\tilde{A} \in \widetilde{\mathrm{Ann}_c}(S)$ a trivialization

$$T_{\tilde{A}} : \underbrace{F_A \xrightarrow{\cong} \mathrm{id}_{\varphi(S)}}_{\text{topologically trivial}}$$

$$\widetilde{\mathrm{Ann}_c}(S) \longrightarrow \mathrm{End}(\mathcal{U}(\lambda))$$
$$\tilde{A} \longmapsto \mathcal{U}(T_{\tilde{A}}) \circ Z_A$$

is a holomorphic map.

$\mathrm{Ann}(S)$ = semigroup of annuli

$$= \left\{ \begin{array}{l} \text{complex cobordism } A \\ \text{with thin parts} \\ \varphi_{in} : S \to \partial_{in} A \\ \varphi_{out} : S \to \partial_{out} A \end{array} \;\middle|\; \begin{array}{c} \partial_{in} A \to A \\ \partial_{out} A \to A \\ \text{are homotopy} \\ \text{equivalent} \end{array} \right\} \Big/ \text{diffeo}$$

and $\widetilde{\mathrm{Ann}_c}$ is the central extension

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \oplus \mathbb{Z} \arrow[r] \arrow[d, "(e^{cz}_{1}, n)"'] & \widetilde{\mathrm{Ann}_c}(S^1) \arrow[r] \arrow[d] & \mathrm{Ann}(S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C}^\times \oplus \mathbb{Z} \arrow[r] & \mathrm{Ann}_c(S^1) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0
\end{tikzcd}$$