Cobordism Hypothesis

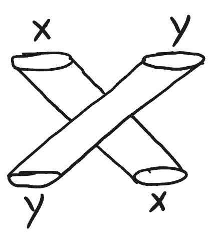
Reference: Higher-Dimensional Algebra and Topological Quantum Field Theory, John C. Baez and James Dolan 1995

Cobordisms

The category of n-cobordisms nCob consists of:

- objects: compact oriented $(n-1)$-mflds M;
- morphisms: oriented cobordisms $\Sigma : M \longrightarrow M'$, so compact oriented n-mflds with $\partial\Sigma = \overline{M} \sqcup M'$, up to diffeomorphism;
- composition: gluing along boundaries;

- identities: $id_M = M \times [0,1]$;
- symmetric monoidal: disjoint union $\sqcup$ and braiding $B_{x,y}$

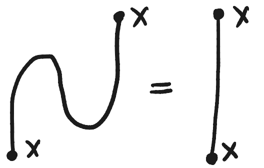


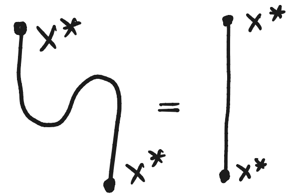
- rigidity: duals are mfld with opposite orientation

unit: $i_x : 1 \to x \otimes x^*$ counit: $e_x : x \otimes x^* \to 1$

$$\begin{tikzcd} x \arrow[rr, "1_x"] \arrow[dr, "i_x \otimes 1_x"'] & & x \\ & x \otimes x^* \otimes x \arrow[ur, "1_x \otimes e_x"'] & \end{tikzcd} \qquad \begin{tikzcd} x^* \arrow[rr, "1_{x^*}"] \arrow[dr, "1_{x^*} \otimes i_x"'] & & x^* \\ & x^* \otimes x \otimes x^* \arrow[ur, "e_x \otimes 1_{x^*}"'] & \end{tikzcd}$$

triangle identities





A TQFT is a rigid symmetric monoidal functor

$$Z : n\mathrm{Cob} \to \mathrm{Vect}. \quad \text{(Atiyah)}$$

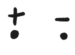
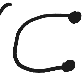
Z is unitary if it preserves a second kind of duality

$$\dagger : n\mathrm{Cob} \to n\mathrm{Cob}$$

which is the identity on objects, and takes each cobordism $f : x \to y$ to the orientation reversed cobordism $f^\dagger : y \to x$. We have

$$1_x^\dagger = 1_x \qquad (fg)^\dagger = g^\dagger f^\dagger.$$

Example: $n=1$, $Z: 1\mathrm{Cob} \longrightarrow \mathrm{Vect}$

- $Z(\emptyset) = \mathbb{C}$
- $Z($  $) = \mathrm{id}_{X^*}$
- $Z($  $) = X \otimes X^*$
- $Z($  $) = \mathbb{C} \xrightarrow{1} X \otimes X^*$
- $Z($  $) = \mathrm{id}_X$
- $Z($  $) = X \otimes X^* \xrightarrow{tr} \mathbb{C}$

$Z($  $) = Z($  $)\, Z($  $)$

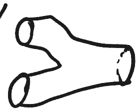
$$= \mathbb{C} \xrightarrow{1} X \otimes X^* \xrightarrow{tr} \mathbb{C}$$

$$\underbrace{\qquad\qquad\qquad}_{\dim X}$$

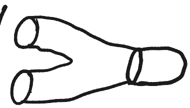
only invariant of a fin. dim. vectorspace $\longrightarrow$ $\dim X$

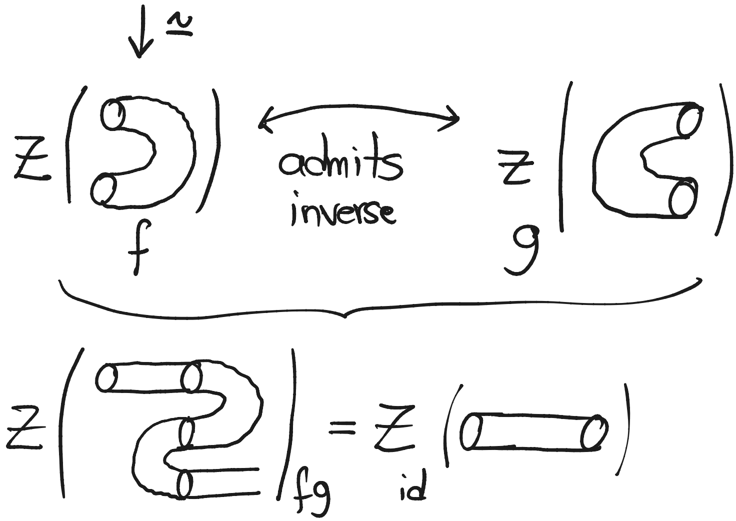
Example: $n=2$ $\iff$ commutative Frobenius algebras

- Generated by one object S^1
- Product:

$Z($  $) = m: A \otimes A \longrightarrow A$

- Identity: $Z($  $) = i: \mathbb{C} \longrightarrow A$
- Trace: $Z($  $) = tr: A \longrightarrow \mathbb{C}$ $\longleftarrow$ nondegenerate
- Nondegenerate pairing:

$Z($  $) = A \otimes A \longrightarrow A \longrightarrow \mathbb{C}$



We can also consider nCob as an n-category:

- objects: compact oriented 0-mflds;
- 1-morphisms: oriented cobordisms of 0-mfds; with boundary
- 2-morphisms: oriented cobordisms of 1-mfds; with corners

  ⋮
- n-morphisms: oriented cobordisms of (n-1)-mflds.

An extended TQFT is a rigid symmetric monoidal functor of n-categories

$$Z : n\mathrm{Cob} \longrightarrow \mathcal{C}^{\otimes}.$$

Cobordism Hypothesis I: Extended TQFTs are easy (to classify). These are defined (in some sense) by their value on a point $Z(*)$. ⟶ 2 objections: frames and duals.

What do we mean by n-category?

Strict n-categories

Let K be a monoidal category. A category C is enriched over K if:

- For pairs (x,y) of objects in C there is an object $\hom(x,y)$ in K;
- For triples (x,y,z) of objects in C there is a morphism

$$\hom(x,y) \otimes \hom(y,z) \longrightarrow \hom(x,z)$$

in K.

Example: Vect enriched over Vect.

A strict 2-category is a category enriched over Cat. A strict $(n+1)$-category is a category enriched over nCat, with a generalized cartesian product of n-categories.

Let C be a 2-category. There are two ways to compose 2-morphisms:

- Vertical composition:

$f, g, h : x \longrightarrow y$

$$\left.\begin{array}{l} \alpha : g \longrightarrow h \\ \beta : f \longrightarrow g \end{array}\right\} \alpha\beta : f \to h$$

composition in K

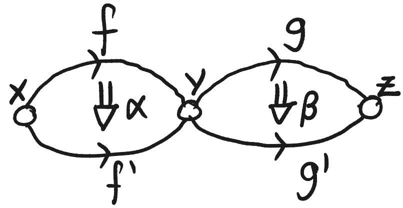
$$\begin{tikzcd} x \arrow[rr, bend left=60, "f"] \arrow[rr, "g" description] \arrow[rr, bend right=60, "h"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\beta"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\alpha"] & & y \end{tikzcd}$$

- Horizontal composition:

$f, f' : x \longrightarrow y$

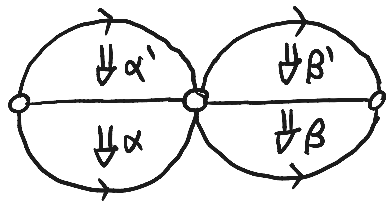
$g, g' : y \longrightarrow z$

$\left.\begin{array}{l} \alpha : f \longrightarrow f' \\ \beta : g \longrightarrow g' \end{array}\right\} \alpha \otimes \beta : gf \Rightarrow g'f'$



- Exchange identity:

$(\alpha\alpha') \otimes (\beta\beta') = (\alpha \otimes \beta)(\alpha' \otimes \beta')$



defines a unique 2-morphism

For n-categories, one can glue n-morphisms in many ways.

Now we want to consider weak n-categories. The main idea of weakening is:

$$\text{equations} \longrightarrow \begin{array}{c}\text{isomorphisms} \\ + \\ \text{coherence laws}\end{array}.$$

Example:

$\boxed{\text{Monoid}}$		$\boxed{\text{Monoidal category}}$
associativity $(xy)z = x(yz)$	$\longrightarrow$	associator $A_{x,y,z}$ pentagon identity
commutativity $xy = yx$	$\longrightarrow$	braiding $B_{x,y}$ $B_{y,x} B_{x,y} = 1_{x \otimes y}$

In weak n-categories, for $k < n$:

equation of k-morphisms $\longrightarrow$ natural $(k+1)$ isomorphism

Strictification theorems:

$n=0$: sets
$n=1$: categories $\Big\}$ $\longrightarrow$ no weakening

$n=2$: bicategories $\longrightarrow$ strict 2-categories

$n=3$: tricategories $\longrightarrow$ semistrict 3-categories

enriched over 2Cats
with weak monoidal product

$$\begin{tikzcd} \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd} \quad C$$

$$D \begin{tikzcd} \circ \arrow[d] \\ \circ \end{tikzcd} \qquad \begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\ \circ \arrow[r] \arrow[ur, phantom, "\Swarrow"] & \circ \arrow[r] \arrow[ur, phantom, "\Swarrow"] & \circ \end{tikzcd}$$

What happens for $n > 3$?

Suspension

Let C be an $(n+k)$-category with one object, one morphism, ..., one $(k-1)$-morphism. Then we get an n-category $\tilde{C}_k$ by re-indexing:

$$\begin{array}{ccc}
\tilde{C}_K & & C \\
\text{objects} & \longrightarrow & \text{morphisms} \\
\text{morphisms} & \longrightarrow & \text{2-morphisms} \\
n\text{-morphisms} & \longrightarrow & (n+1)\text{-morphisms}
\end{array}$$

and adquire properties

$$\begin{array}{ccc}
C & & \tilde{C}_K \\
\text{category} & \longrightarrow & \text{monoid} \\
\text{(strict/weak) 2-category} & \longrightarrow & \text{(resp.) monoidal category} \\
\text{(strict/weak/semistrict) 3-category} & \longrightarrow & \text{(resp.) monoidal 2-category} \\
(n+k)\text{-category} & \xrightarrow{K\text{-times}} & K\text{-tuply monoidal } n\text{-category}
\end{array}$$

	$n=0$	$n=1$	$n=2$
$K=0$	sets	categories	2-categories
$K=1$	monoids	monoidal categories	monoidal 2-categories
$K=2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$K=3$	"	symmetric monoidal categories	weakly involutory monoidal 2-categories
$K=4$	"	"	strongly involutory monoidal 2-categories
$K=5$	"	"	"

Why is the entry $n=0, k=2$ a commutative monoid?
Eckmann-Hilton argument:

- Let $1 = 1_{1_x}$. We want to show $\hom(1_x, 1_x)$ is commutative.

Using that $\alpha = 1 \otimes \alpha = \alpha \otimes 1$ we have

$$\begin{aligned} \alpha \otimes \beta &= (1\alpha) \otimes (\beta 1) = (1 \otimes \beta)(\alpha \otimes 1) \\ &= \beta\alpha = (\beta \otimes 1)(1 \otimes \alpha) \\ &= (\beta 1) \otimes (1 \alpha) = \beta \otimes \alpha \end{aligned}$$

Conversely, a commutative monoid is a 2-category with one object and one 1-morphism.

In general we have

	$n=0$	$n=1$	$n=2$
$k=0$	sets	categories	2-categories
$k=1$	xy	$x \otimes y$	$x \otimes y$
$k=2$	$xy = yx$	$B_{x,y}: x \otimes y \to y \otimes x$	$B_{x,y}: x \otimes y \to y \otimes x$
$k=3$	"	$B_{x,y} = B_{y,x}^{-1}$	$I_{x,y}: B_{x,y} \Rightarrow B_{y,x}^{-1}$
$k=4$	"	"	$I_{x,y} = (I_{y,x}^{-1})^{-1}_{hor}$

Let $S: n\mathrm{Cat}_{k-1} \to n\mathrm{Cat}_k$ be left-adjoint to the forgetful functor $F: n\mathrm{Cat}_k \to n\mathrm{Cat}_{k-1}$. Call this the suspension functor.

Stabilization Hypothesis: $S : n\mathrm{Cat}_k \longrightarrow n\mathrm{Cat}_{k+1}$ is an equivalence of categories for $k \geqslant n+2$.

Motivations arise from homotopy theory:

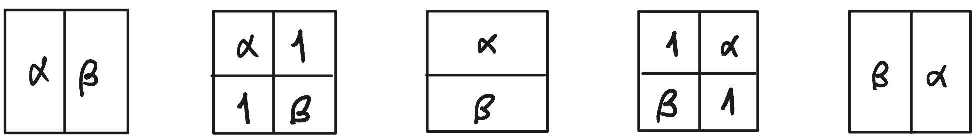
- The n-th fundamental grupoid $\Pi_n(X)$ is a sort of weak n-category;
- $n=0$ column is familiar from fundamental group $\pi(X)$

$\pi_0(X) \longrightarrow$ set

$\pi_1(X) \longrightarrow$ group

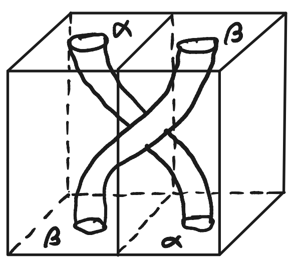
$\pi_2(X) \longrightarrow$ abelian group

Familiar proof from Eckmann-Hilton argument



$\alpha \otimes \beta$ $\quad (1 \otimes \beta)(\alpha \otimes 1)$ $\quad \beta\alpha$ $\quad (\beta \otimes 1)(1 \otimes \alpha)$ $\quad \beta \otimes \alpha$

These resemble frames of a movie capturing a braiding.



Suspension functor S inherits its name from topological suspension functor which gives sequence of homotopy classes

$$[X,Y] \xrightarrow{S} [SX,SY] \xrightarrow{S} [S^2X,S^2Y] \to \dots$$

which stabilizes for $k \geq n+2$ (gives isomorphisms)

Recall that in the category nCob there exist two distinct dualities:

- category nCob : $\underset{\text{objects}}{X \to X^*}$ $\qquad$ $\underset{\text{bordisms}}{f \to f^\dagger}$

$\downarrow$

- n-category nCob : $n+1$ distinct levels of duality.

Appropriate n-category of which TQFT are representations should be a k-tuply monoidal n-category with duals. For $0<j<n$ there should be units and counits

$$i_f : 1_Y \to ff^*, \quad e_f : f^*f \to 1_X$$

satisfying some weak triangle identity. Furthermore,

$$f^{**} = f, \quad (fg)^* = g^*f^*.$$

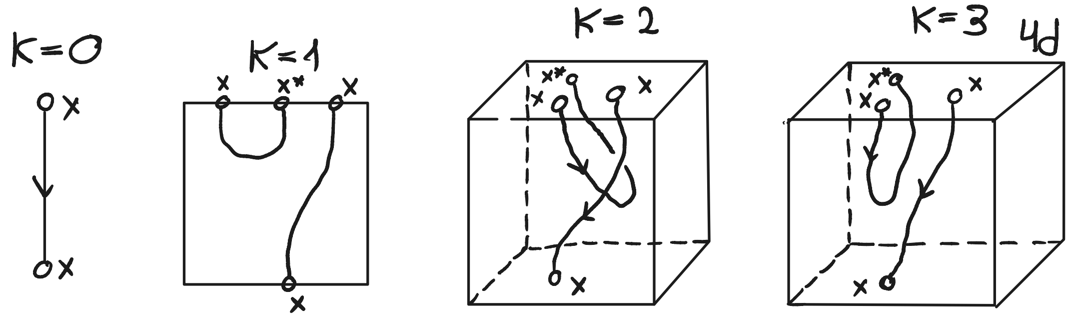
Let $C_{n,k}$ be the free semistrict k-tuply monoidal n-category with duals on one object.

Known examples for $n=3$ suggest we should consider framed manifolds and cobordisms.

Tangle Hypothesis: The n-category of framed n-tangles in $n+k$ dimensions is $(n+k)$-equivalent to the free weak k-tuply monoidal n-category with duals on one object.

- n-tangle in $n+k$ dimensions: n-mfld with corners embedded in $[0,1]^{n+k}$ such that codim j corners are mapped into subset with last j-coordinates 0 or 1.

Example: $n=1$

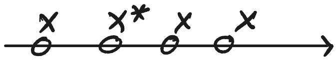


- n-tangles in dim $n+k$ form an n-category:
  - objects: finite subsets (mflds?) of $[0,1]^k$.
  - 1-morphisms: class of 1-tangles in $[0,1]^{k+1}$, going from classes of 0-tangles on $[0,1]^k \times \{0\}$ to classes on $[0,1]^k \times \{1\}$;
  - j-morphisms: class of j-tangles in $[0,1]^{j+k}$, going from classes of $(j-1)$-tangles in $[0,1]^{j+k-1} \times \{0\}$ to classes on $[0,1]^{j+k-1} \times \{1\}$;
  - composition: vertical stacking and rescalling of cubes;
  - tensor product: justaposition of cubes;
  - duality: reflection of j-tangles along last coordinate axis.

Example: $n=0$.

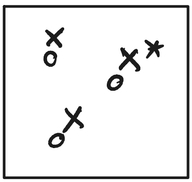
- $k=0 \longrightarrow$ set with duals $\{x, x^*\}$
- $k=1 \longrightarrow$ monoid with involution

noncommuting words



$\Longleftrightarrow \quad x x^* x x \ldots$

- $k=2 \longrightarrow$ commutative monoid with involution



extra dim $\Rightarrow$ commutative

$\Longleftrightarrow \quad x^p (x^*)^q$

Comparing with results from knot theory confirm other cases: [Turaev, Yetter]

Isotopy classes of framed 1-tangles in 3 dimensions $\Longleftrightarrow$ morphisms of $C_{1,2}$

Tranversality results from differential topology imply for $k \geqslant n+2$, embeddings of compact n-mflds in $\mathbb{R}^{n+k}$ are all isotopic, which supports the stabilization hypothesis.

Stabilization hypothesis + Tangle hypothesis

$\Downarrow$

Cobordism hypothesis II: The n-category of which n-dimensional extended TQFTs are representations is the free stable weak n-category with duals on one object.

$C_{n,k}$ stabilizes for $k \geq n+2$, so call the stable category $C_{n,\infty}$. Here:

- objects: framed 0-mflds;
- 1-morphisms: framed 1-mflds with boundary;
- 2-morphisms: framed 2-mflds with corners;
- All these embedded in $[0,1]^{n+k}$ for $k \geq n+2$. In particular, n-morphisms are isotopy classes of framed n-tangles in $n+k$ dimensions, for $k \geq n+2$.

Cobordism Hypothesis III: An n-dimensional unitary extended TQFT is a weak n-functor, preserving all levels of duality, from the free stable weak n-category with duals on one object to nHilb.

↳ weak n-category of n-Hilbert spaces. Some sort of module category with duals.